算法设计与分析

Lecture 12: Backtracking

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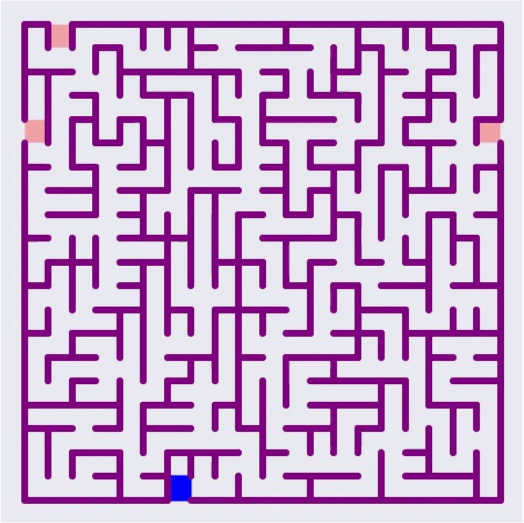
Backtracking

- A simple and straightforward strategy to escape from a maze is:
  - Go as deep as possible until reach a dead end.
  - Go back to the last fork and choose another path.
- If we have a sign at the fork to show dead ends, we can avoid that path to save time.
- This is the idea of backtracking (回溯). It is a refinement of the brute force approach by avoiding dead ends in advance.

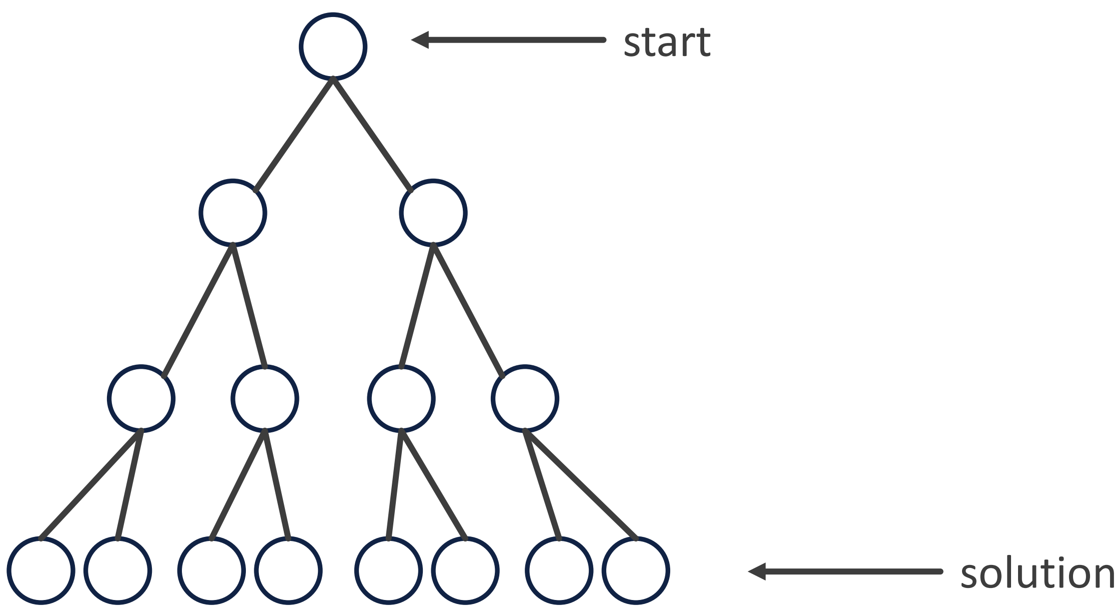


A maze

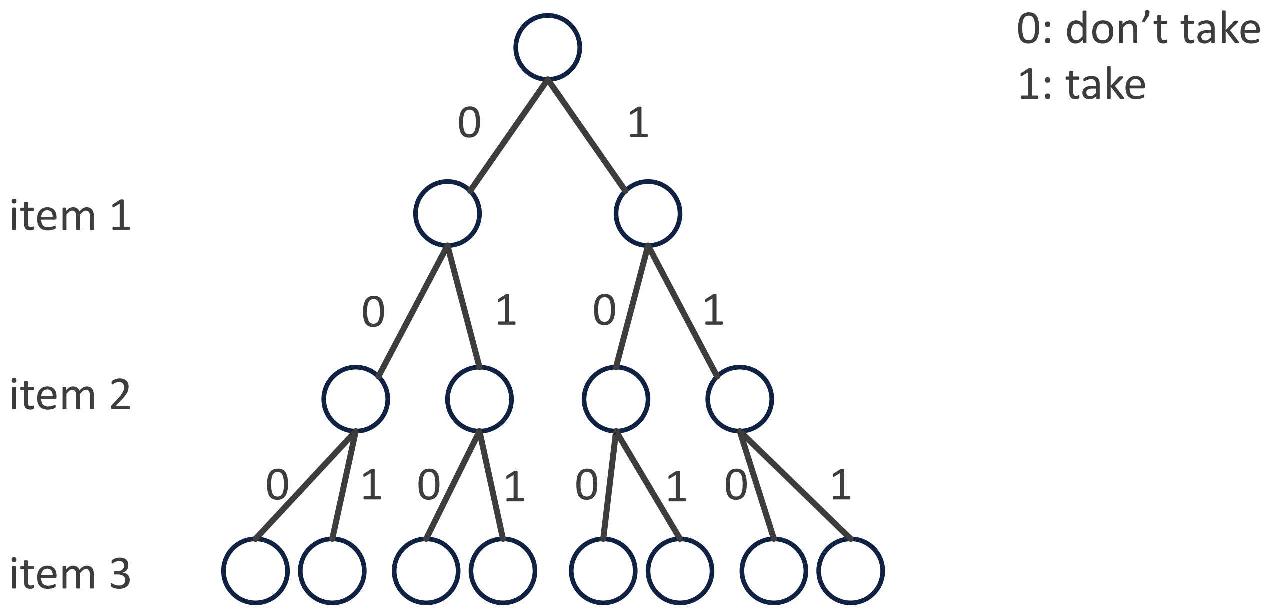
Image source: https://gfycat.com/ru/babyishtightjanenschia-mazesolving-backtracking

Backtracking

- Given the an optimization problem, we usually make a sequence of decisions. It can be represented as a tree.
- We start from the root and the solutions are the leaves.



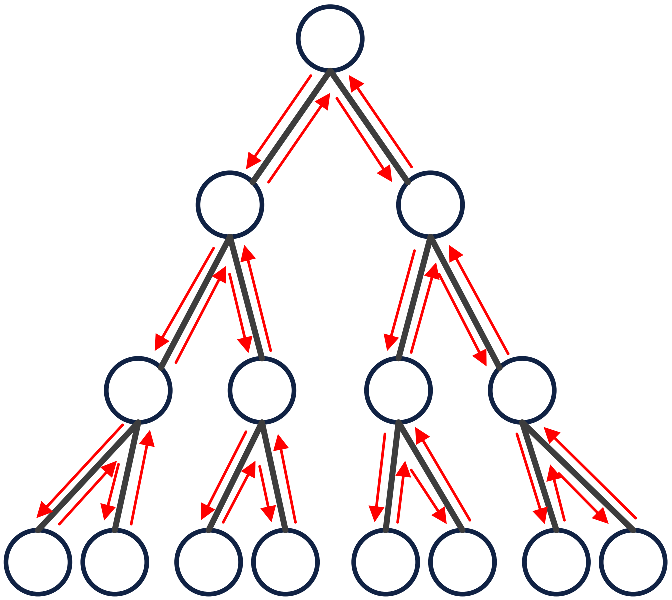
Backtracking

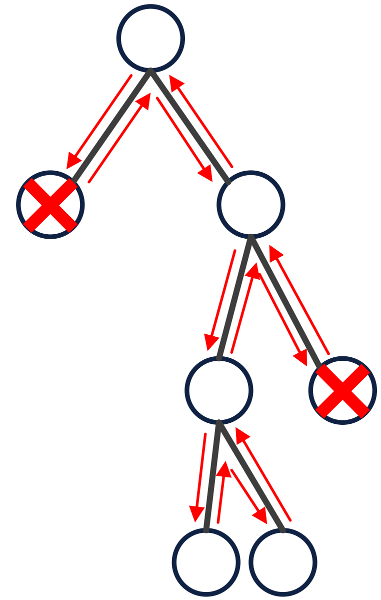


Solution space for 0/1 knapsack problem with 3 items

DFS vs. Backtracking

If we know that going along this branch has no hope, we don't need to try! It will save a lot of time.





Backtracking

- Backtracking is all about HOPE!
- We only continue to search solutions only if there is still hope!

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Image source: https://giphy.com/explore/there-is-still-hope

Backtracking

- In the backtracking method, the solutions are represented by vectors $(x_1, x_2, \dots, x_n)$.
- In step $i + 1$, we start from a partial solution $(x_1, x_2, \dots, x_i)$ and try to extend it by adding another element x_{i+1}.
- After extending it, we will test whether $(x_1, x_2, \dots, x_i, x_{i+1})$ is still possible as a partial solution (check hope).

Backtracking

The steps involved in the backtracking method are:

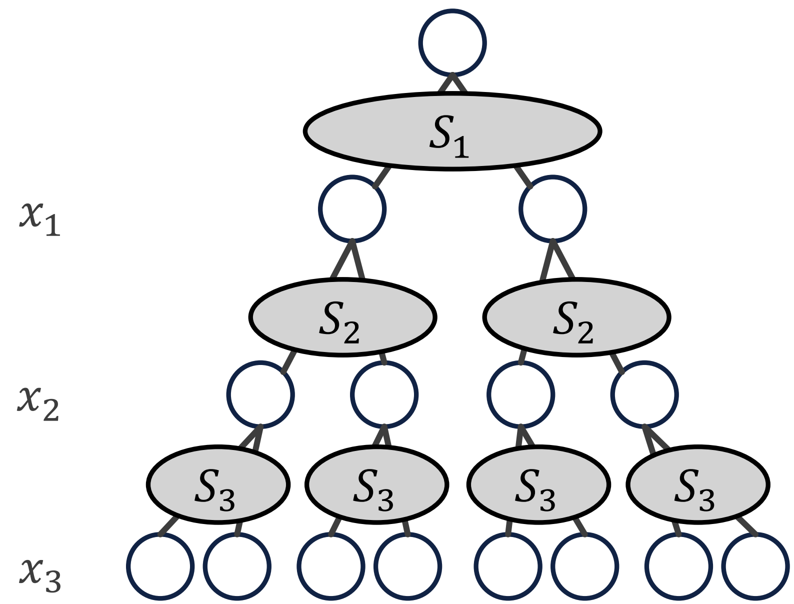
1. Define a solution space (解空间) for the problem. This space must include at least one (optimal) solution to the problem.
   - If S_i is the domain of x_i, then $S_1 \times S_2 \times \cdots \times S_n$ is the solution space of the problem.
   - Generally, the solution space is very huge, so the cost of searching a objective solution are often unimaginable.
   - For backtracking to be efficient, we must prune (剪枝) the search space.

Backtracking

2. Organize the solution space so that it can be searched easily. The typical organization is either a graph or a tree.
3. Searched the solution space in a DFS manner and avoid moving into subspaces that cannot possibly lead to the answer.

Solution Space Tree

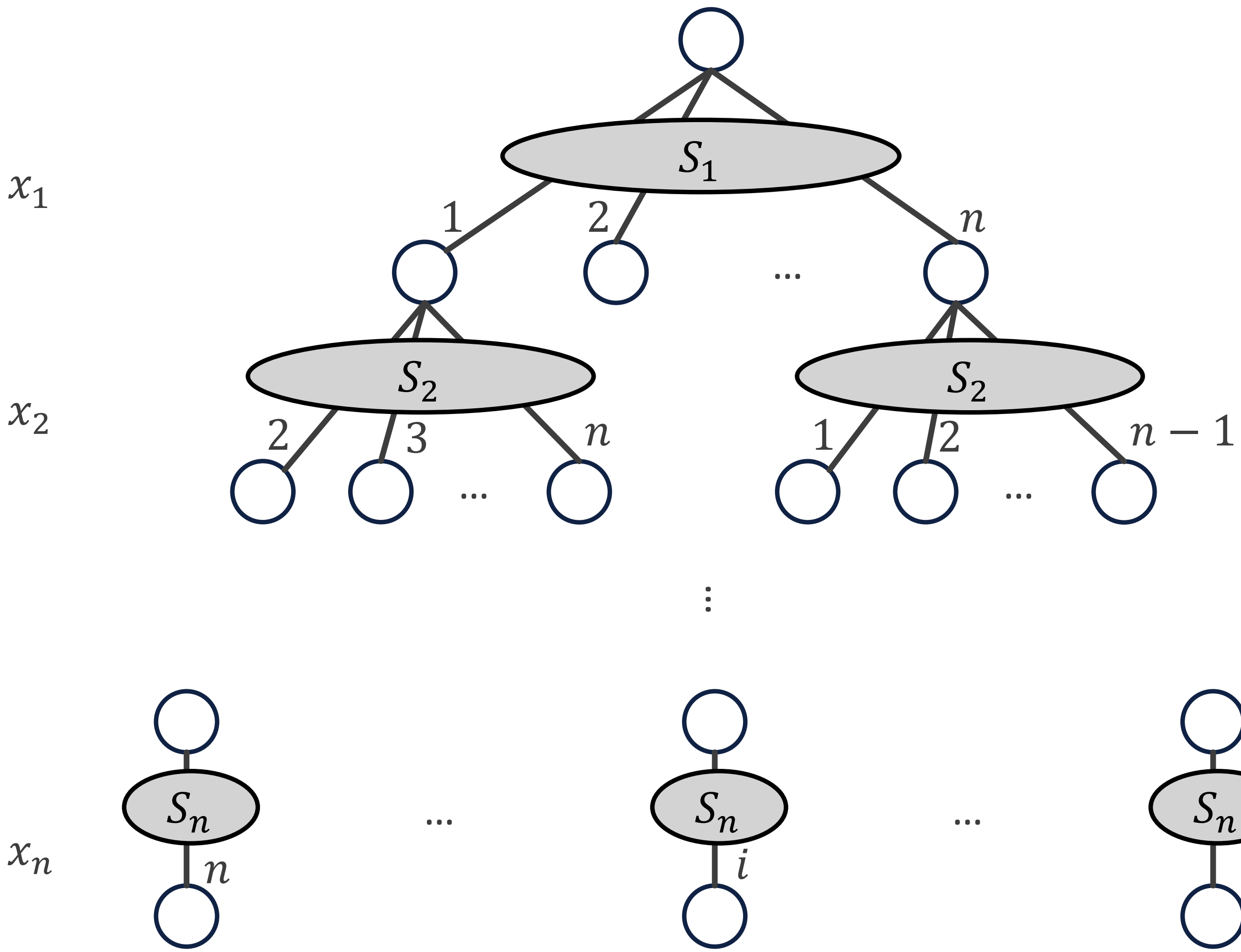
- We set up a tree structure such that the leaves represent members of the solution space.
- So we organize solution space as a solution space tree (解空间树).
- Backtracking can easily be used to iterate through all subsets or permutations of a set.



Example

- When the problem asks for an n-element permutation that optimizes some function, the solution space tree is a permutation tree.
- How many permutations are there of an n-element set?
  - There are n choices for x_1.
  - There are $n - 1$ choices for x_2.
  - …
  - There is only 1 choices for x_n.

S_{i+1} depends on the choice of x_i



General Backtracking Template

```
Backtrack(i)
    if i > n then Update(x)
    else
        for each a ∈ S_i do
            x_i ← a
            if C(i) and B(i) then
                Backtrack(i + 1)
```

Reaching leave means that it is a feasible solution.

Hope checking condition, key of backtracking. Without it, it is just brute-force.

Pruning

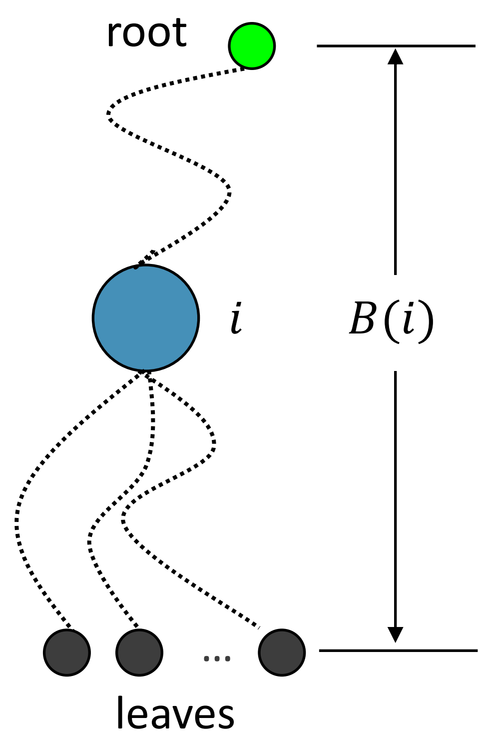
- In backtracking, we have a constraint function (约束函数) $C(i)$ and a bounding function (限界函数) $B(i)$, to prune invalid branches and to focus the search on branches that appear most promising.
  - Keep in mind, we don't waste time on hopeless branches.
- In order to improve the performance of search, applying backtracking requires specifying at least the following three points:
  - How to choose an the constraint function.
  - How to compute upper bounds (for maximum problem) and lower bounds (for minimum problem).
  - How to make use of the constraint function and the bounding function to prune.

Constraint Function

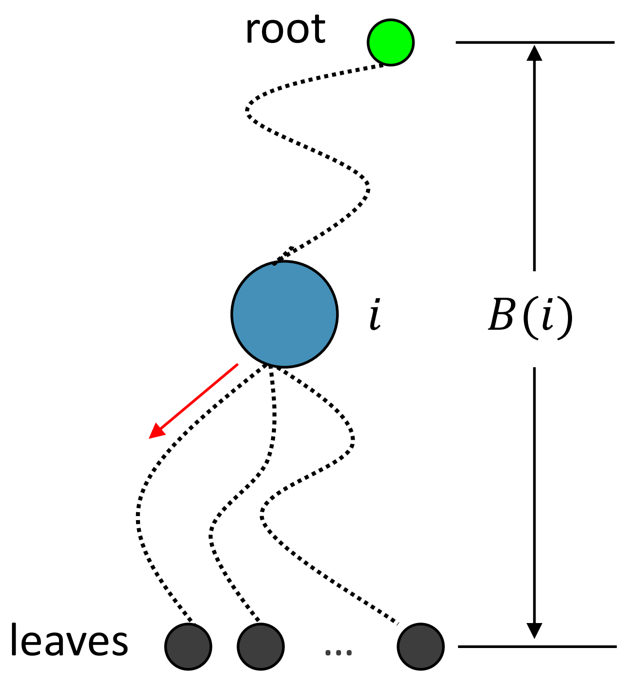
- Constraint function is to check the feasibility of the current solution.
- Usually, it can be easily built by the problem requirement. For example:
  - 0/1 knapsack problem: check if adding the next item exceeds W.
  - Permutation problem: check if the number has been selected.
  - Hamiltonian cycle problem: check (1) if next vertex is connected to the current vertex; (2) if the last vertex is connected to the first vertex; (3) if there exist duplicated vertex in the path.
  - Coloring problem: check if the color for the next vertex is same as its neighbors.

Bounding Function

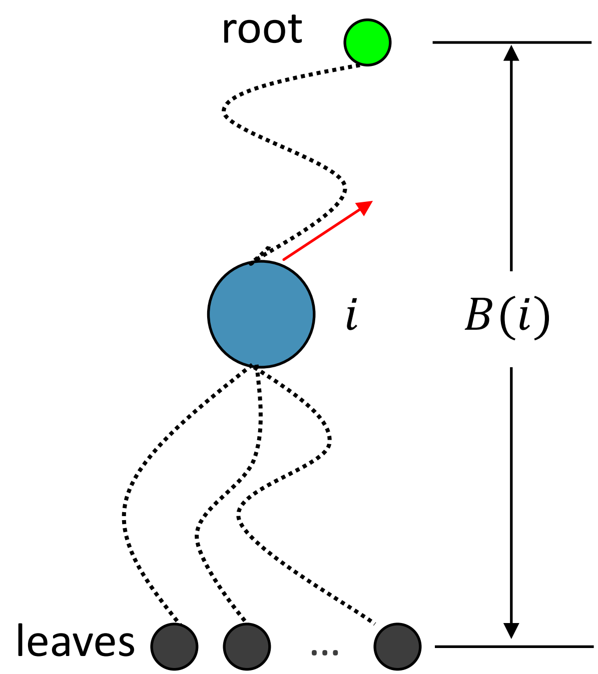
- Bounding function is for optimization problem.
- For maximization problem, it calculates the upper bound of this branch $B(i)$ and compare with the existing best solution $bestc$.
  - If $B(i) > bestc$, there is still hope, keep searching!
  - If $B(i) \leq bestc$, all solutions along this branch will not better than the existing best solution, stop!



Bounding Function



$B(i) > bestc$, go ahead, there is still hope!



$B(i) \leq bestc$, go back, it is hopeless!

CONTAINER LOADING PROBLEM

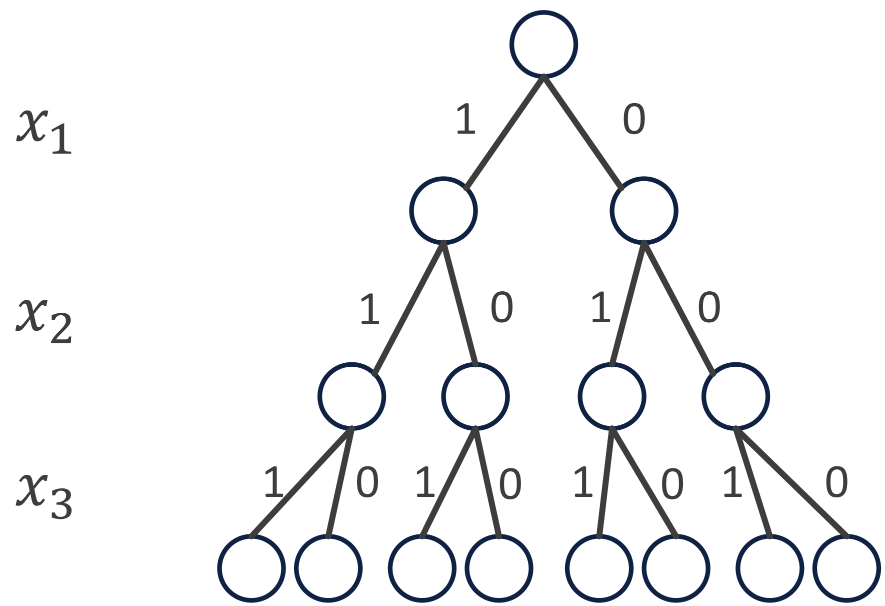
Container Loading Problem

- Given n containers (集装箱), container i has weight w_i. The ship can hold containers of total weight up to W.
- Container Loading problem is to load as many containers as is possible without sinking the ship.
- Assuming that the solutions are represented by vectors $(x_1, x_2, \ldots, x_n)$, where $x_i \in \{0,1\}$. 1 denotes taking container i and 0 denotes not taking container i.
- The container loading problem can be formally stated as follows:

$$\max \sum_{i=1}^{n} w_i x_i \qquad s.t. \sum_{i=1}^{n} w_i x_i \le W$$

Container Loading Problem

- Each x_i has two options to choose: take and not take.
- Therefore, $|S_i| = 2$ and the size of the solution space is 2^n. It also means that the solution space tree has 2^n leaves.



Solution space tree with $n = 3$

Container Loading Problem

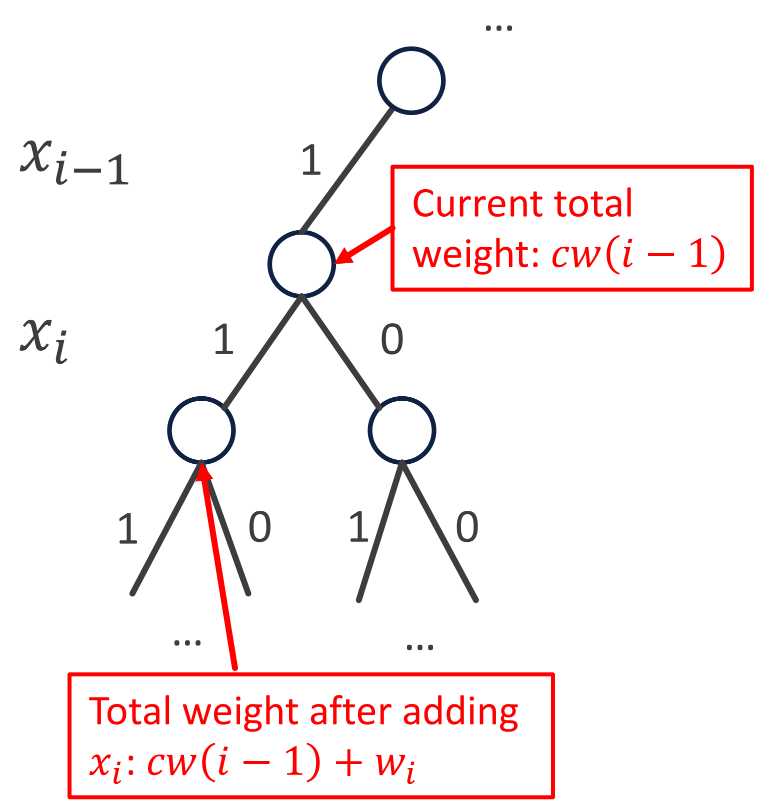
- We first design the constraint function.
- Let $cw(i)$ denote the current weight up to level i, namely

$$cw(i) = \sum_{j=1}^{i} w_j x_j$$

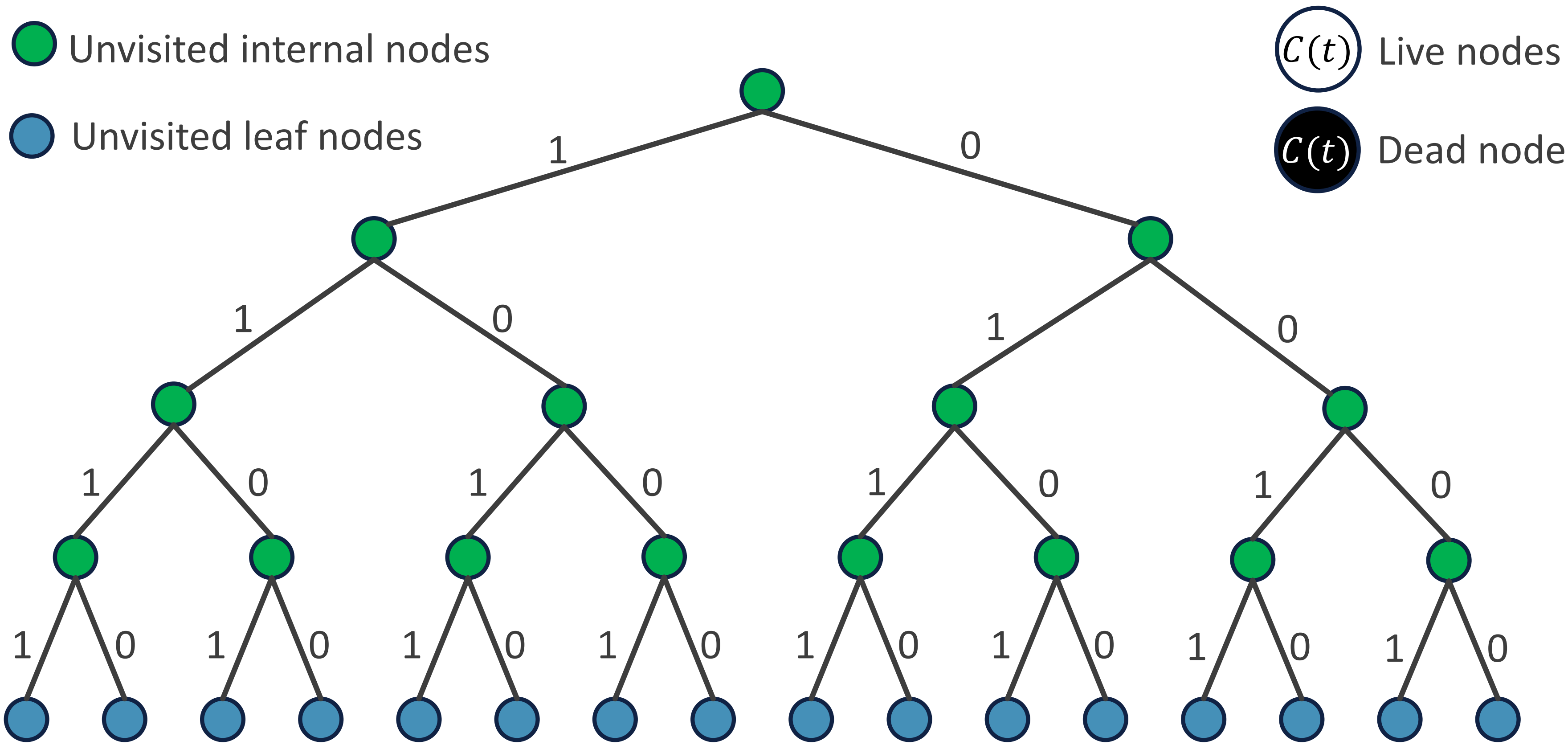
then the constraint function is

$$C(i) = cw(i-1) + w_i$$

- The pruning condition is $C(i) > W$, which means there is no capacity to take container i.

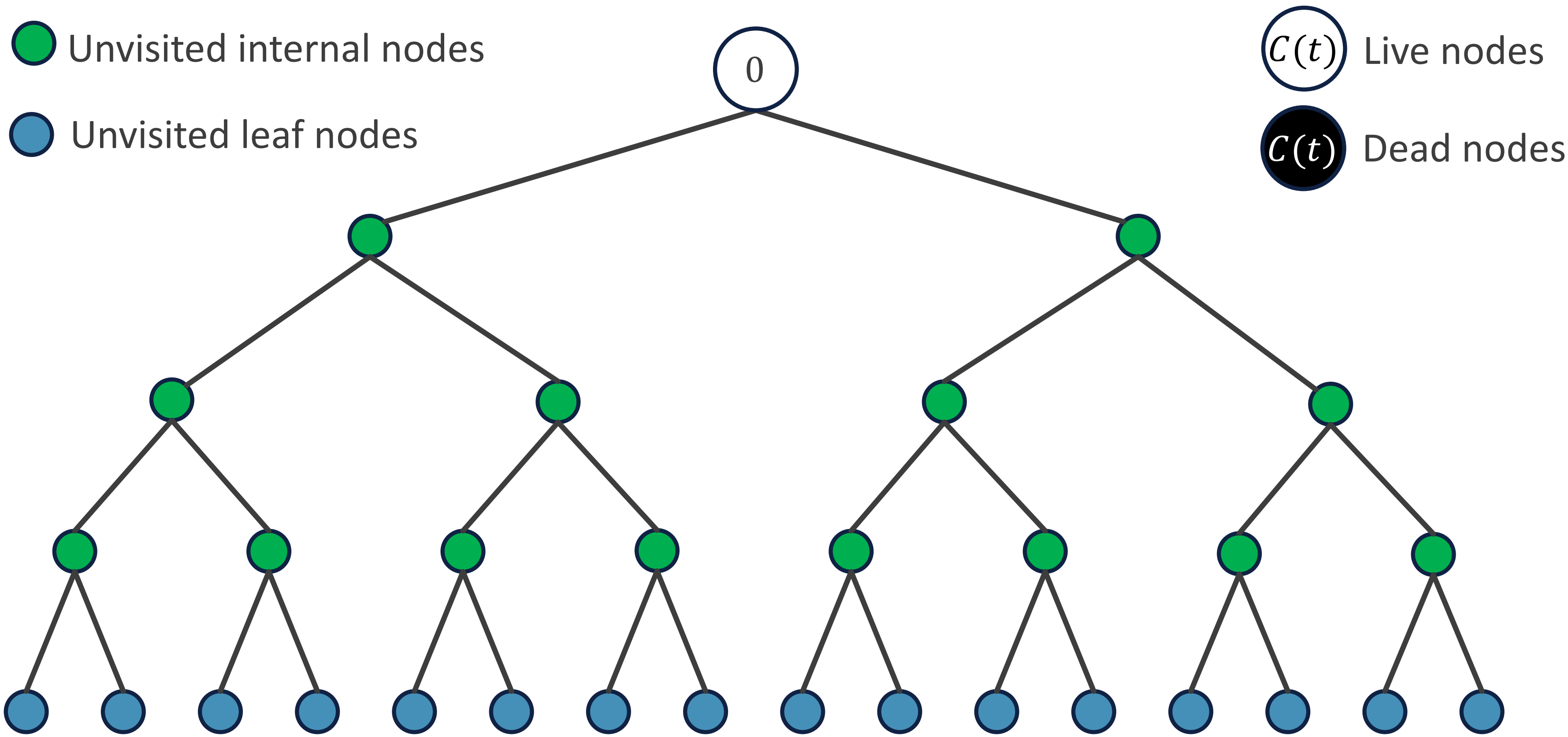


Example



Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

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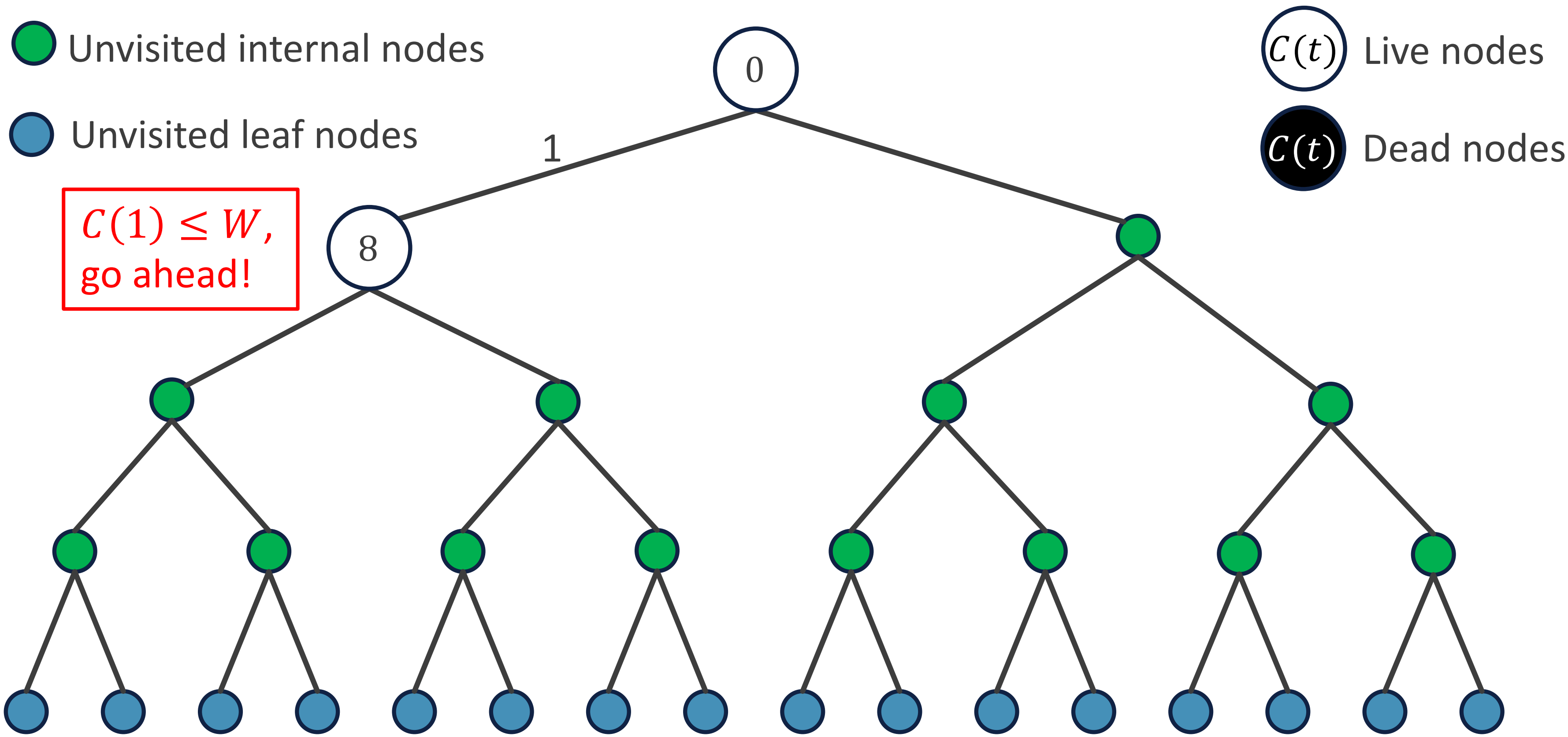

Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

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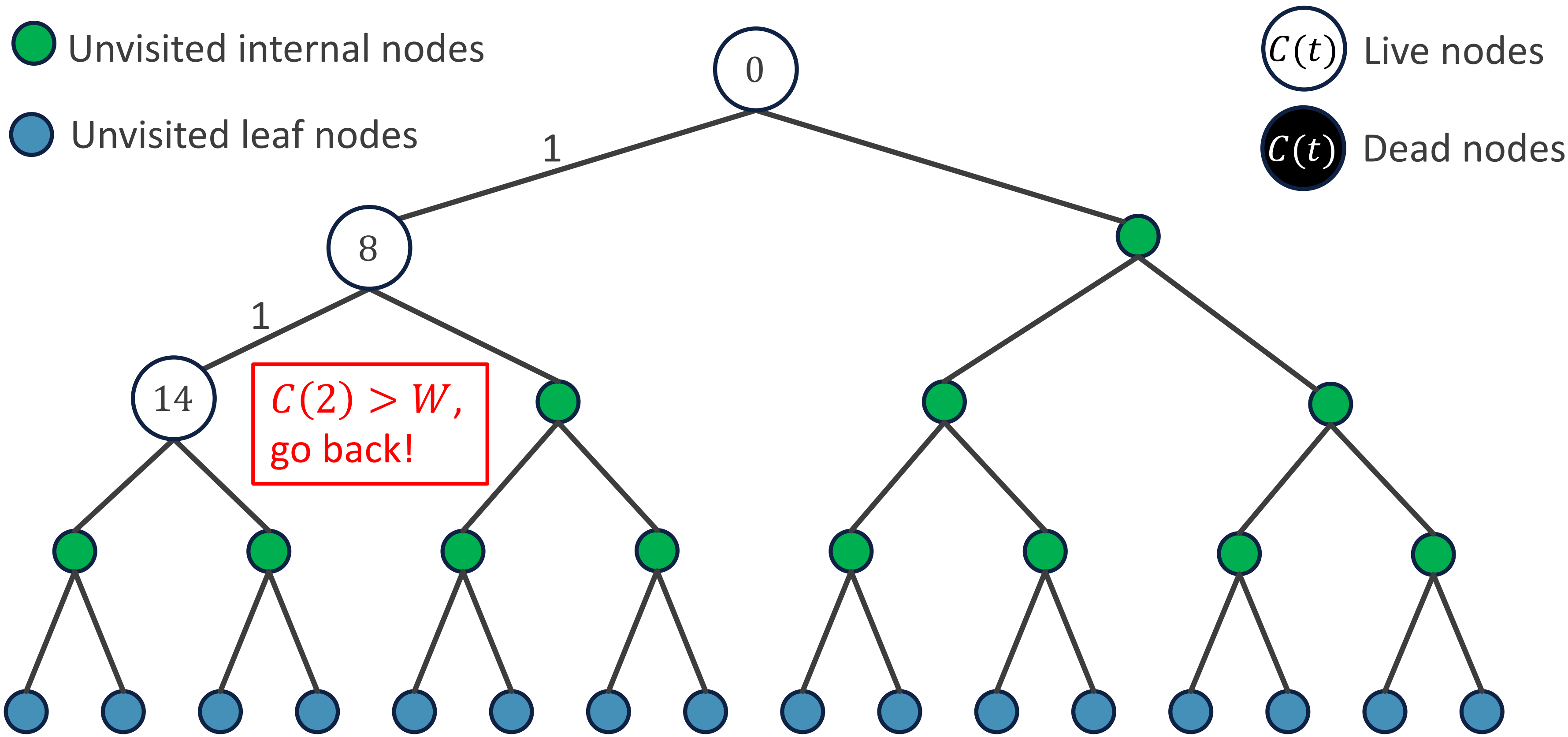

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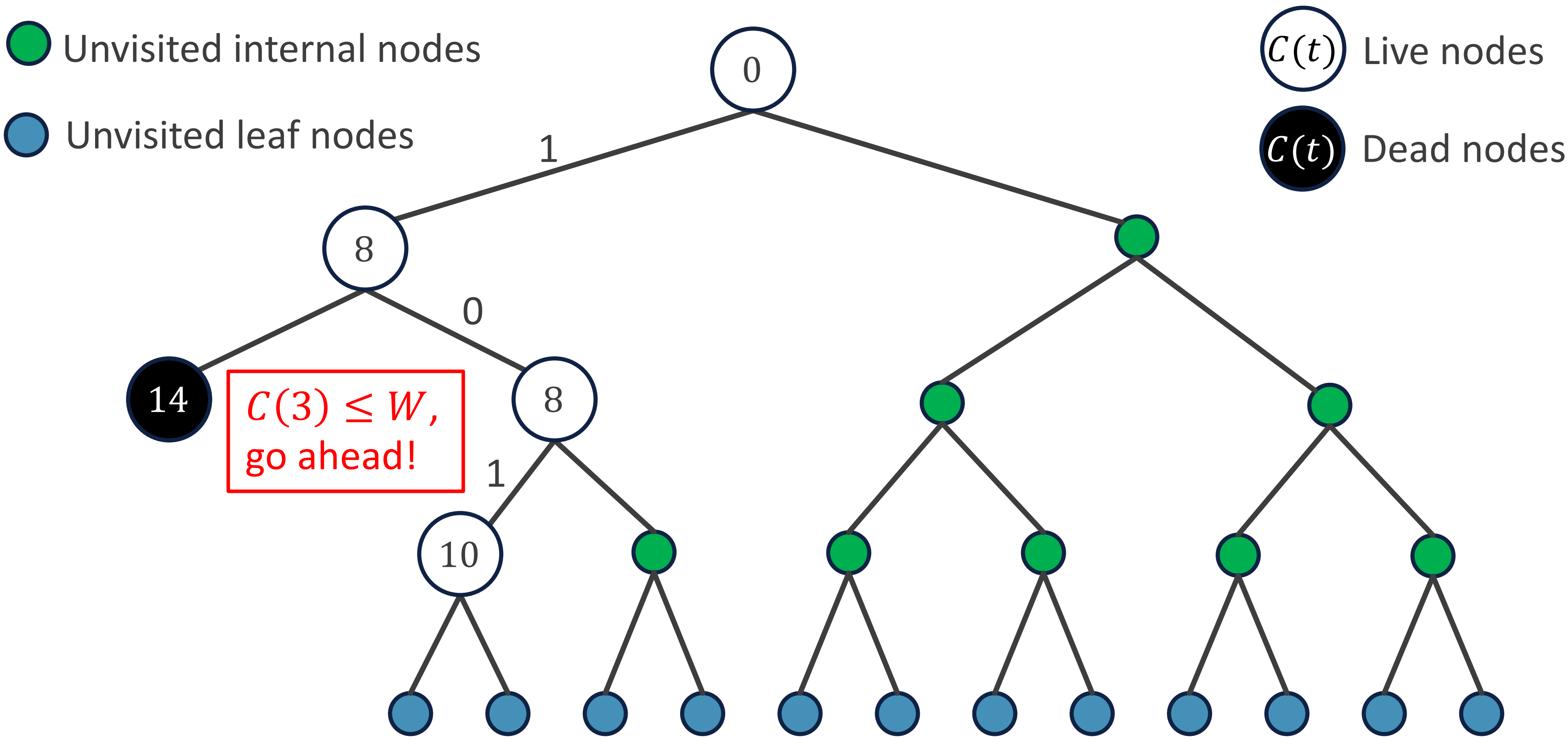
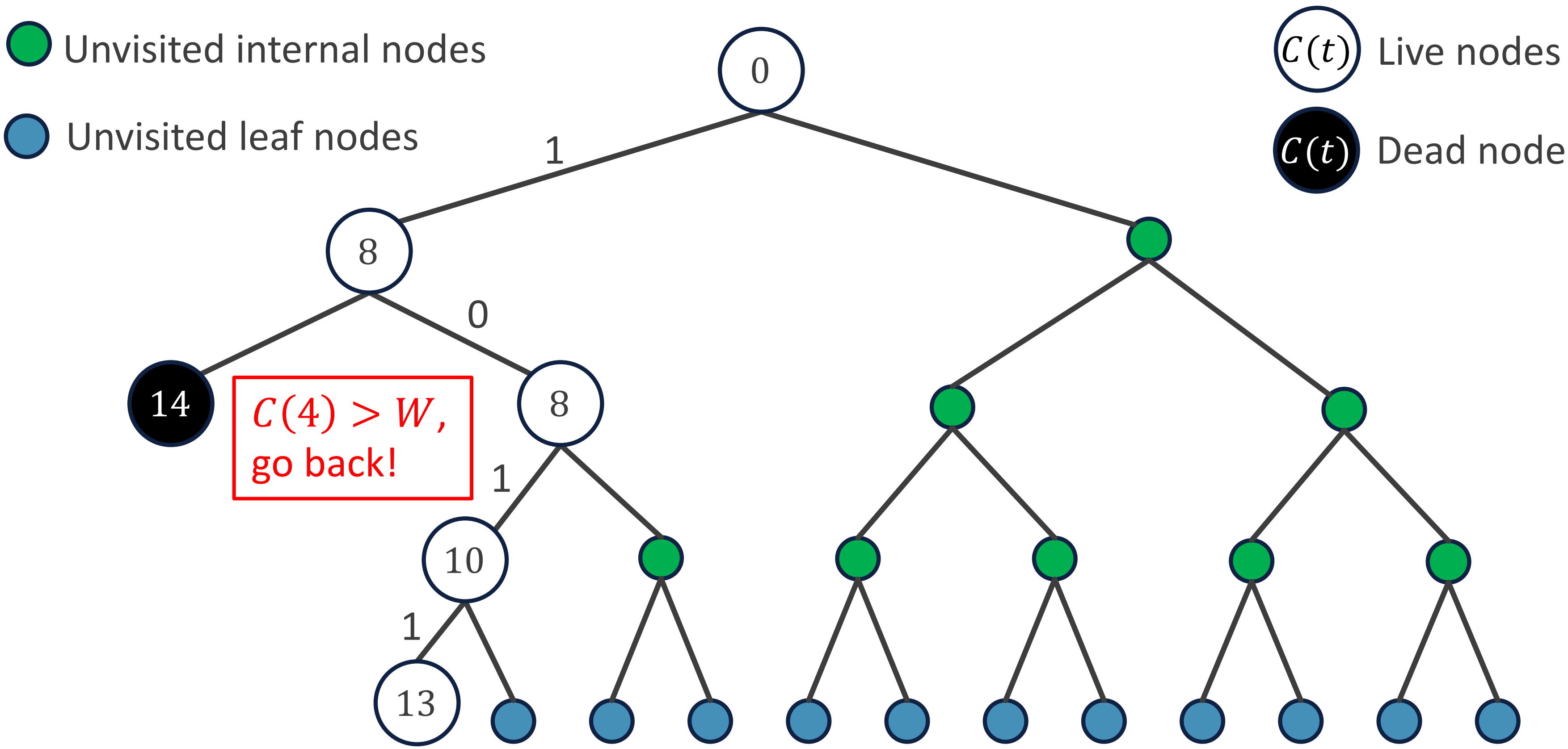
Example



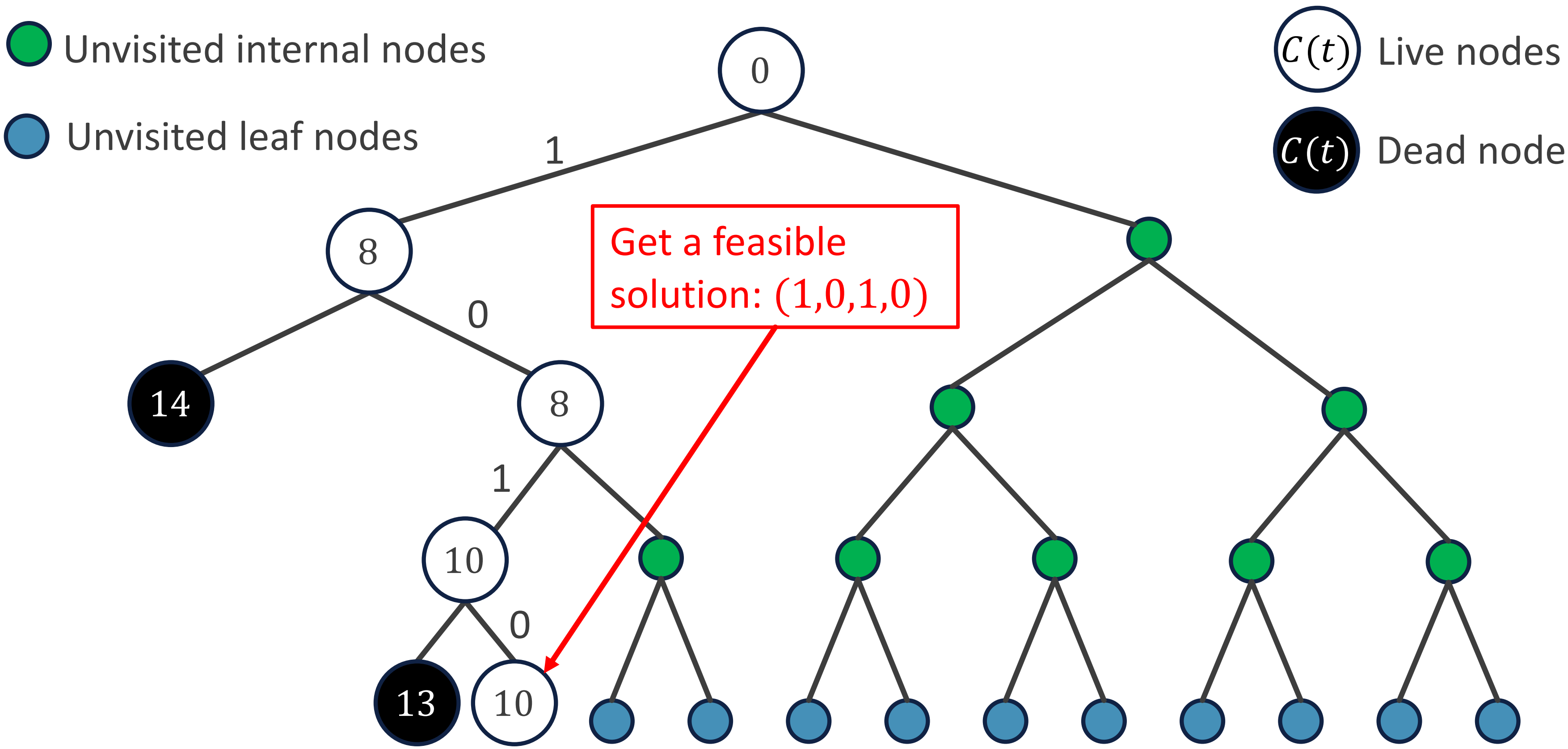
Backtracking for $n = 4, w = [8,6,2,3], W = 12$

Example



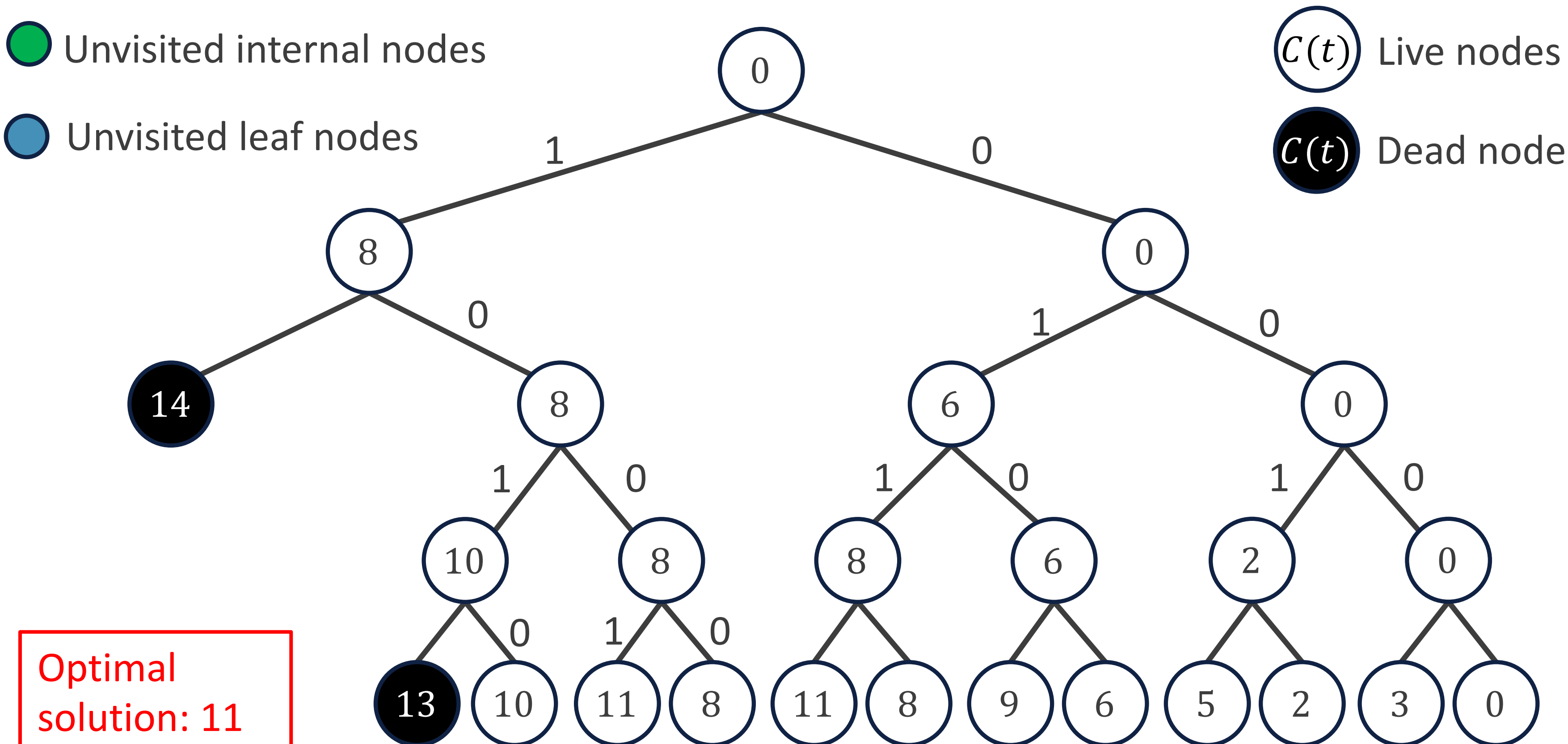
Backtracking for $n = 4, w = [8,6,2,3], W = 12$

Example



Backtracking for $n = 4, w = [8,6,2,3], W = 12$

Example



Optimal solution: 11

Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Pseudocode

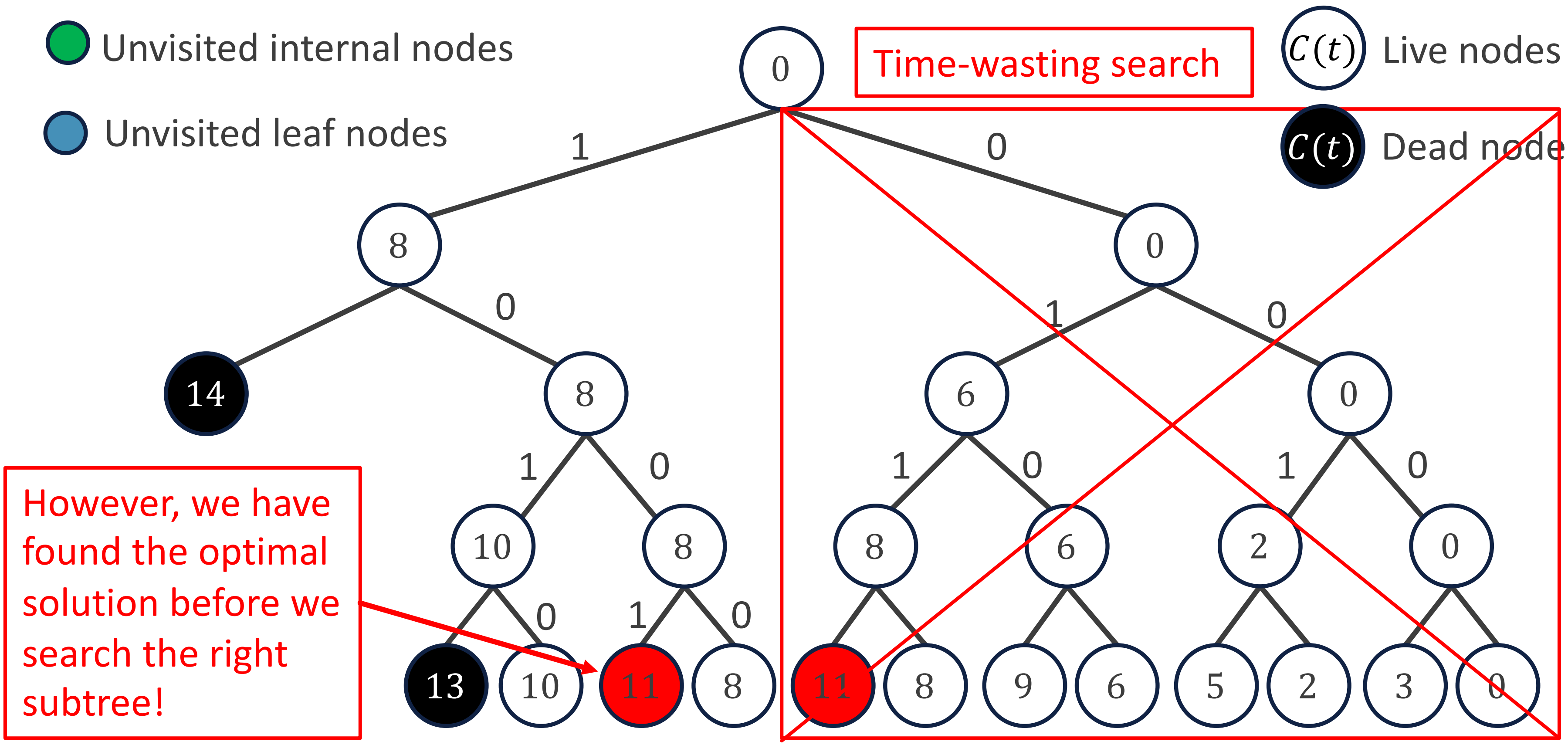
```
BacktrackLoading(i)
  if i > n then
      if cw > bestw then
          bestw ← cw          ← Store best solution so far.
  else
      if C(i) ≤ W then
1         cw ← cw + w[i]
          BacktrackLoading(i + 1)
          cw ← cw − w[i]
0     BacktrackLoading(i + 1)
```

Note: we don't actually build a tree structure. Instead, we simply use recursion.

Go ahead by taking container i.

Subtract the weight of container i before we go back.

Example



Backtracking for $n = 4, w = [8,6,2,3], W = 12$

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Example

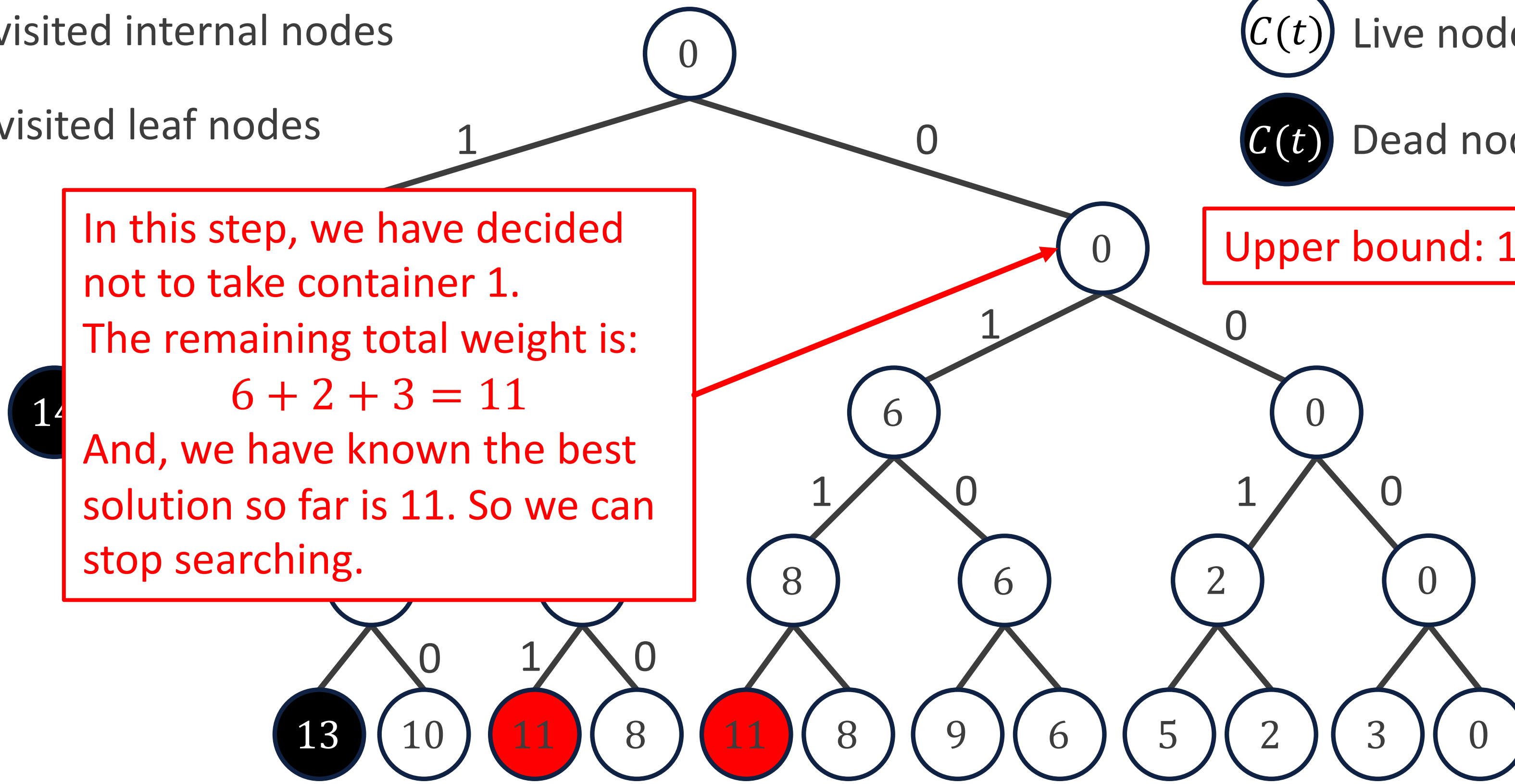
Unvisited internal nodes

Unvisited leaf nodes

$C(t)$ Live nodes

$C(t)$ Dead nodes

Upper bound: 11



In this step, we have decided not to take container 1.
The remaining total weight is:
$$6 + 2 + 3 = 11$$
And, we have known the best solution so far is 11. So we can stop searching.

Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Container Loading Problem

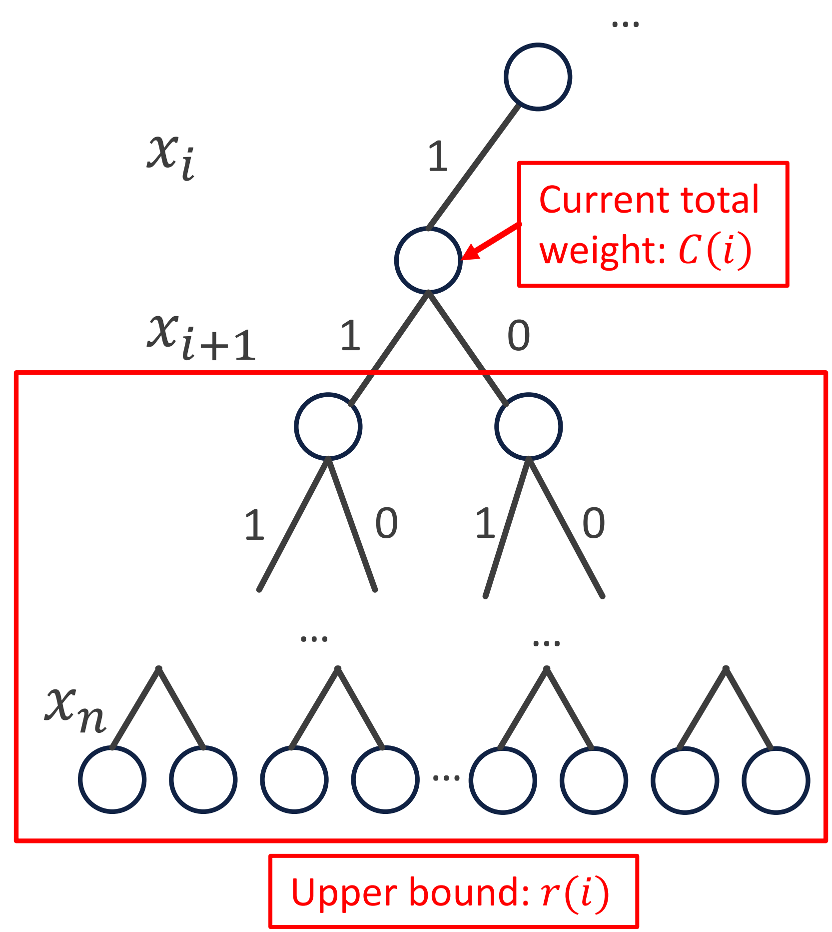
- Now, as an improvement, we add the bounding function:

  $$B(i) = C(i) + r(i)$$

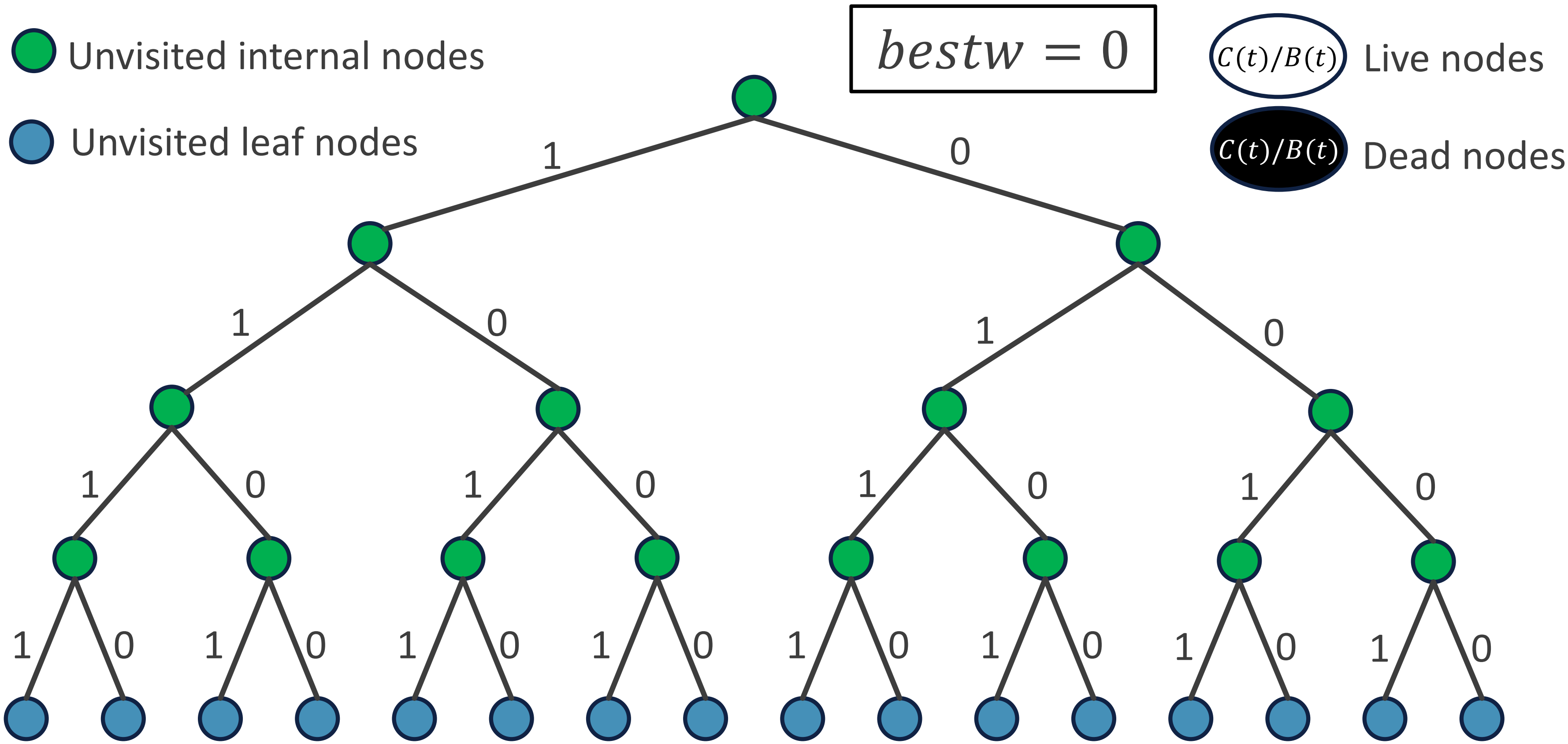
  where, $r(i)$ denotes the weight sum of the remaining containers, namely,

  $$r(i) = \sum_{j=i+1}^{n} w_j$$

- The pruning condition is $B(i) \leq bestw$, which means the continuing searching along this branch will not give better solution.

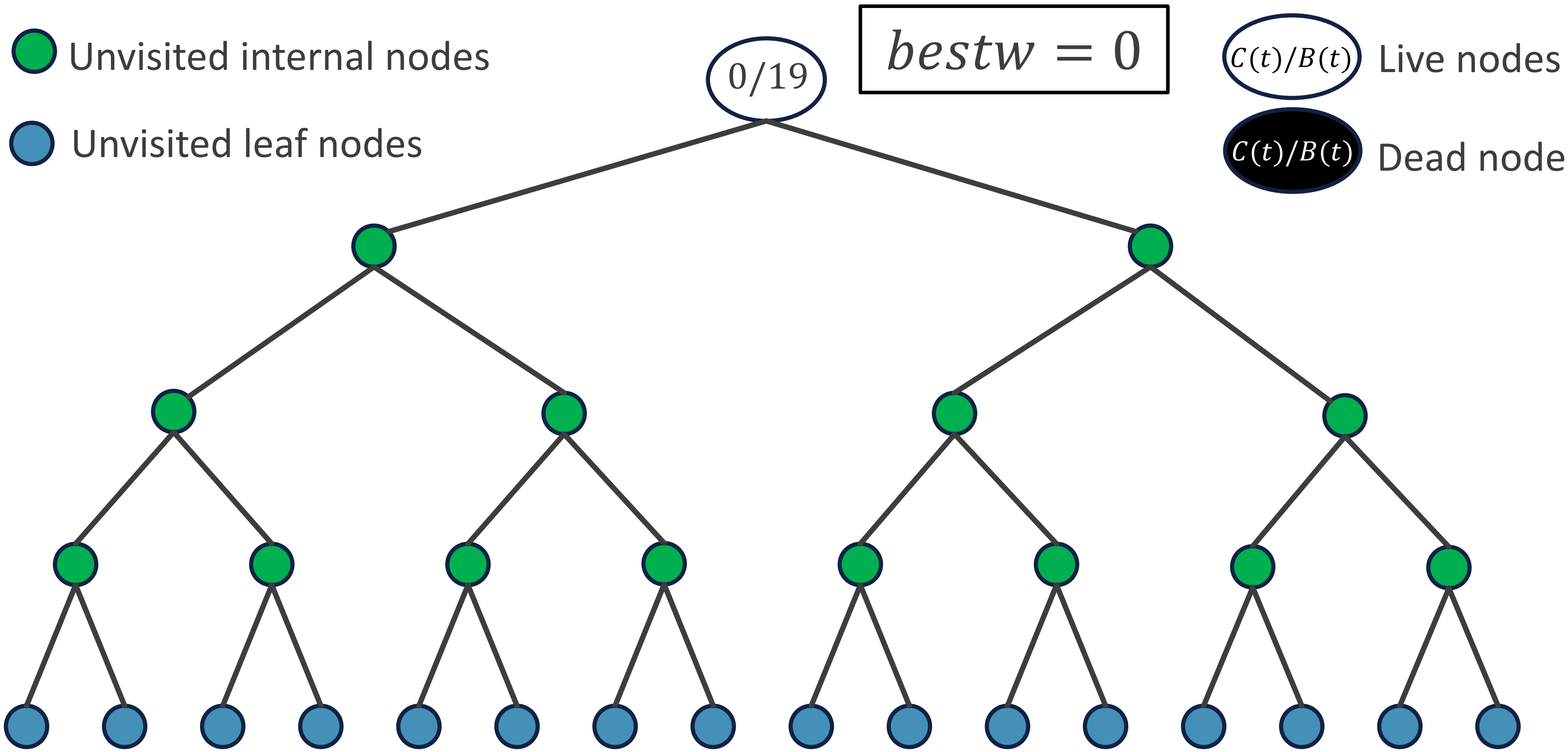


Example



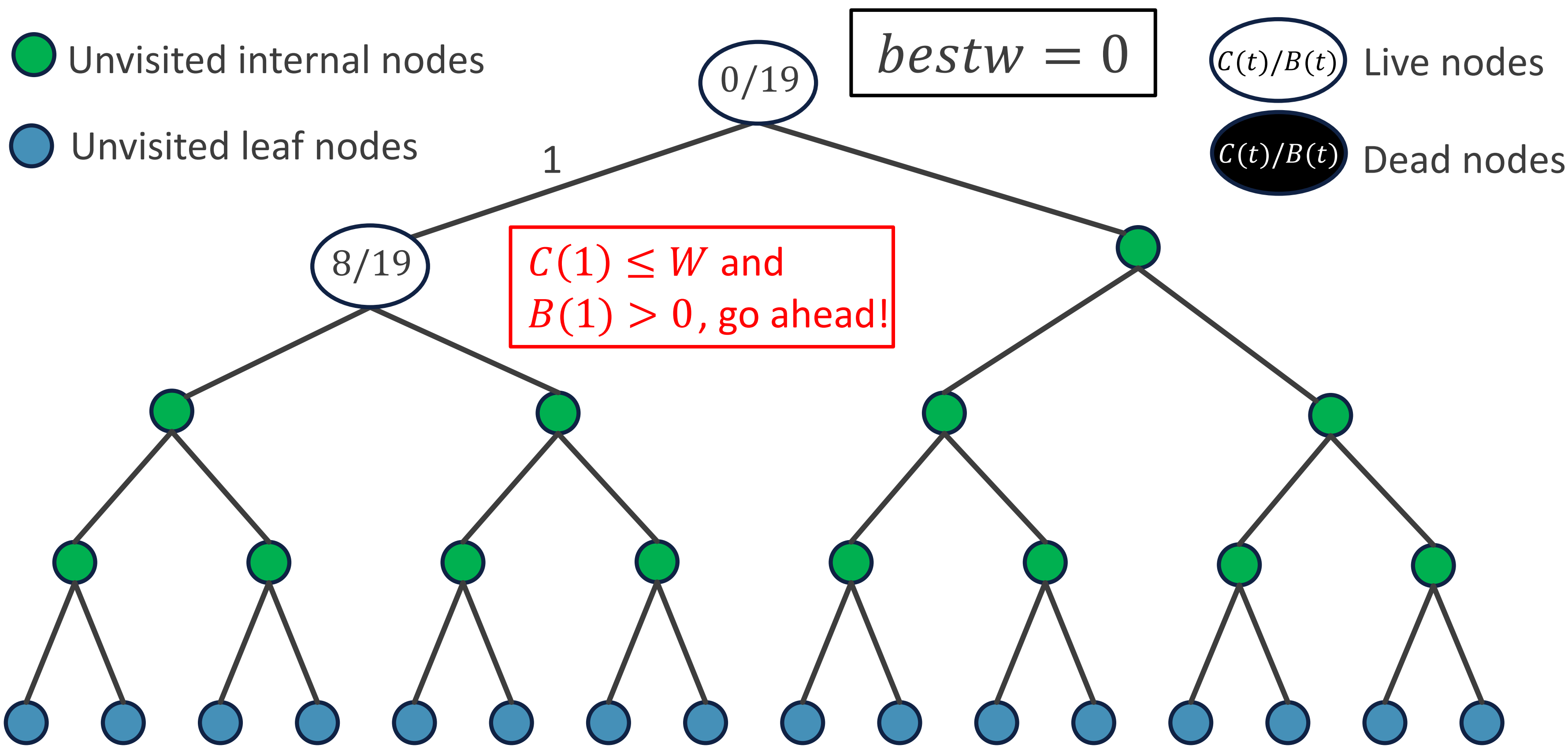
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



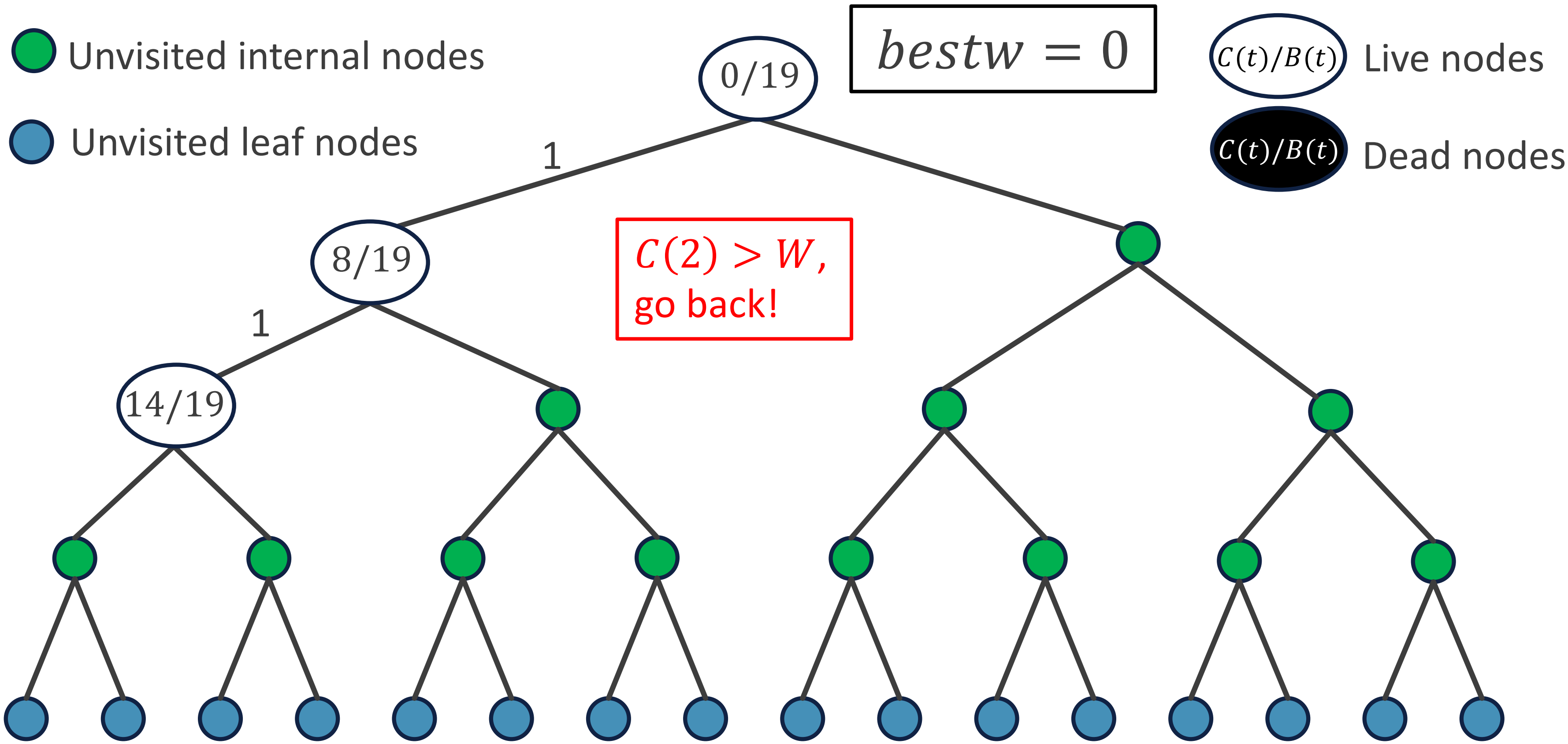
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



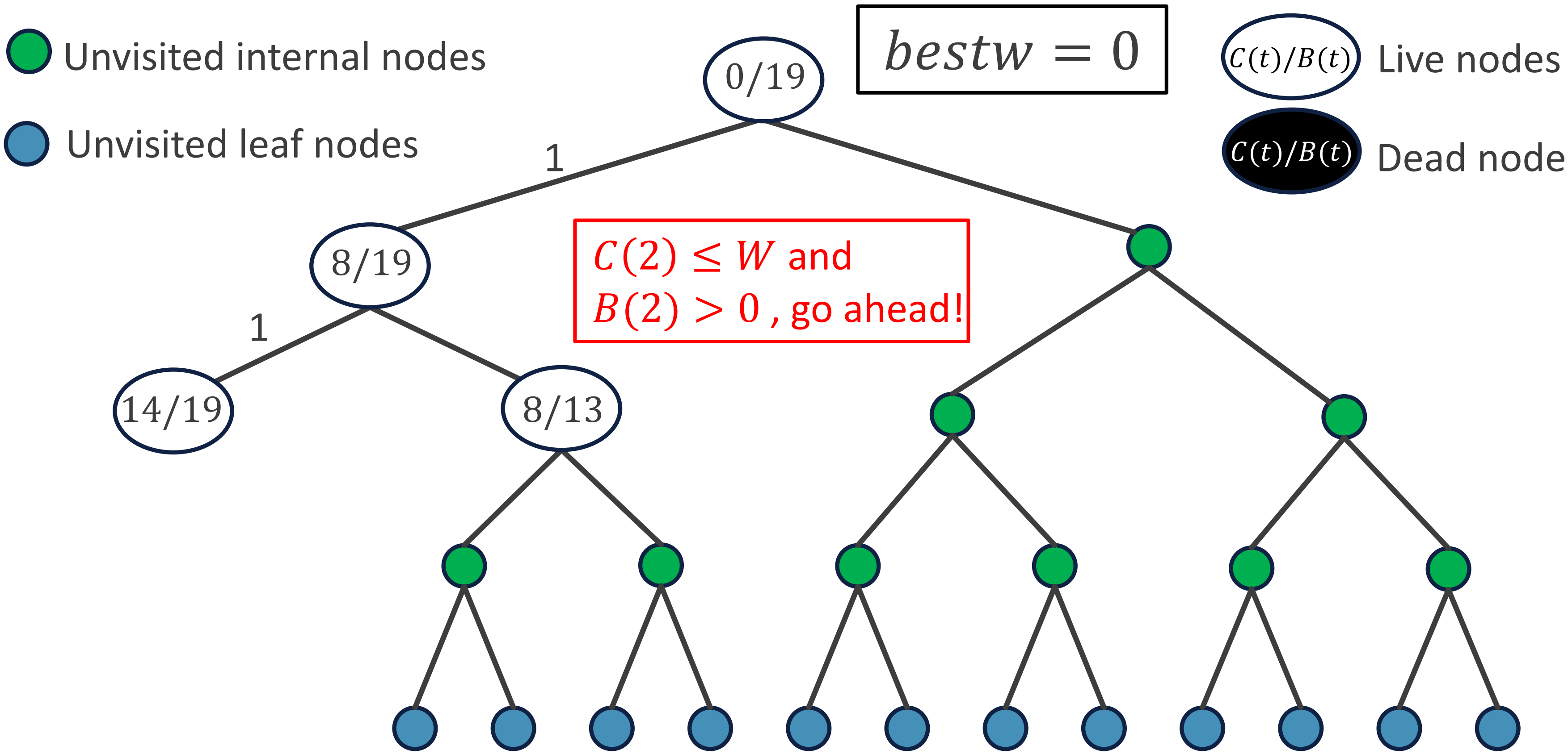
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



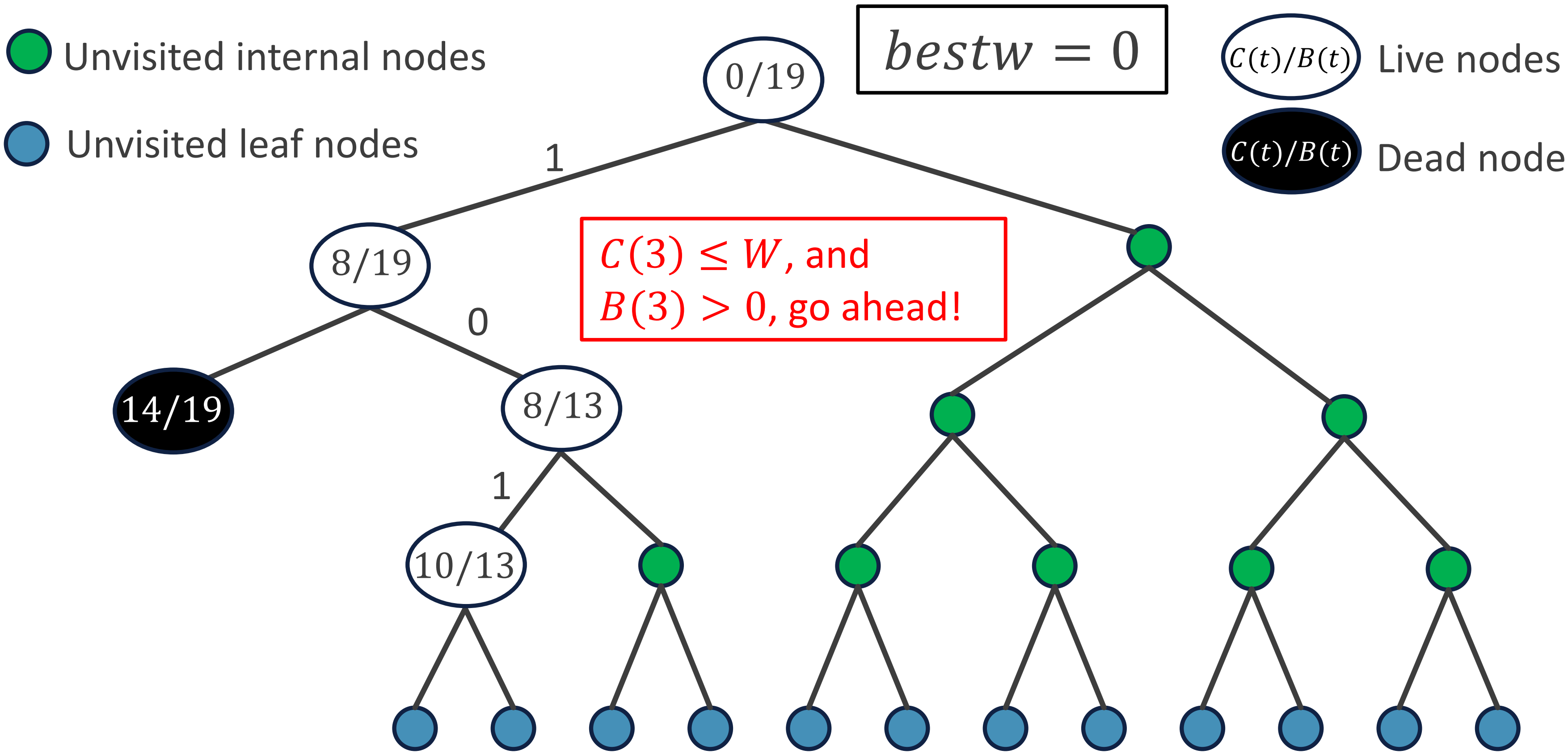
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



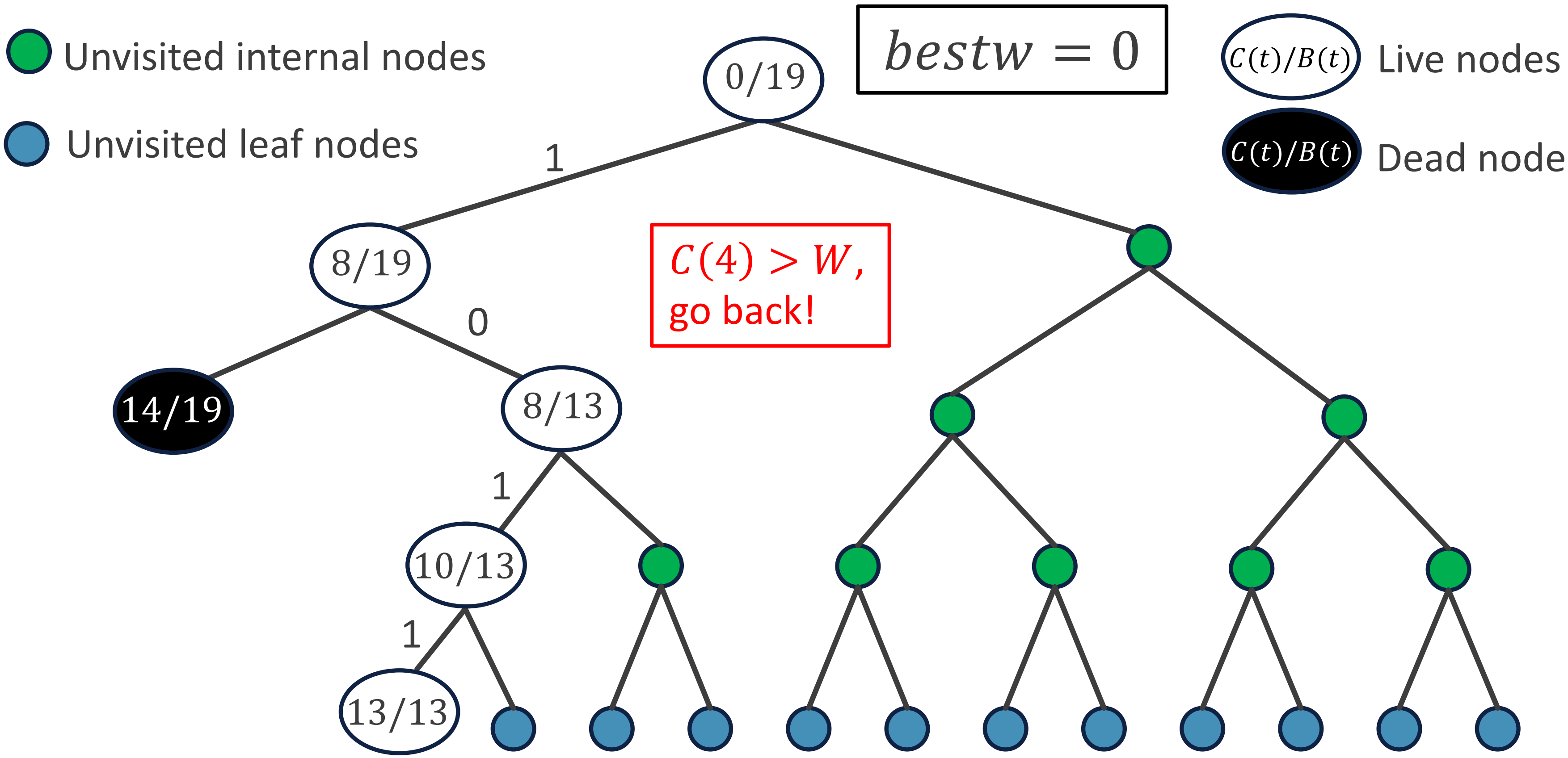
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



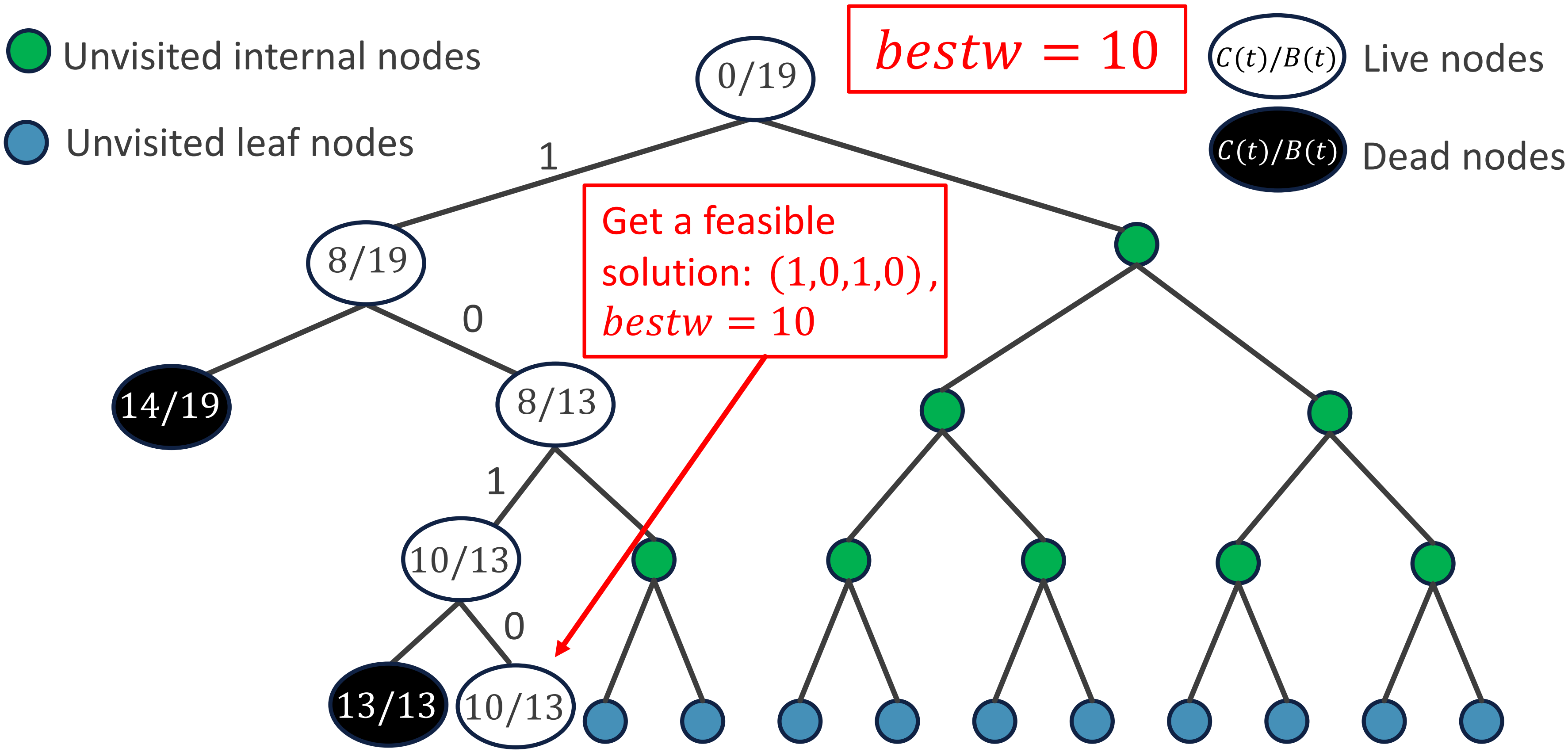
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



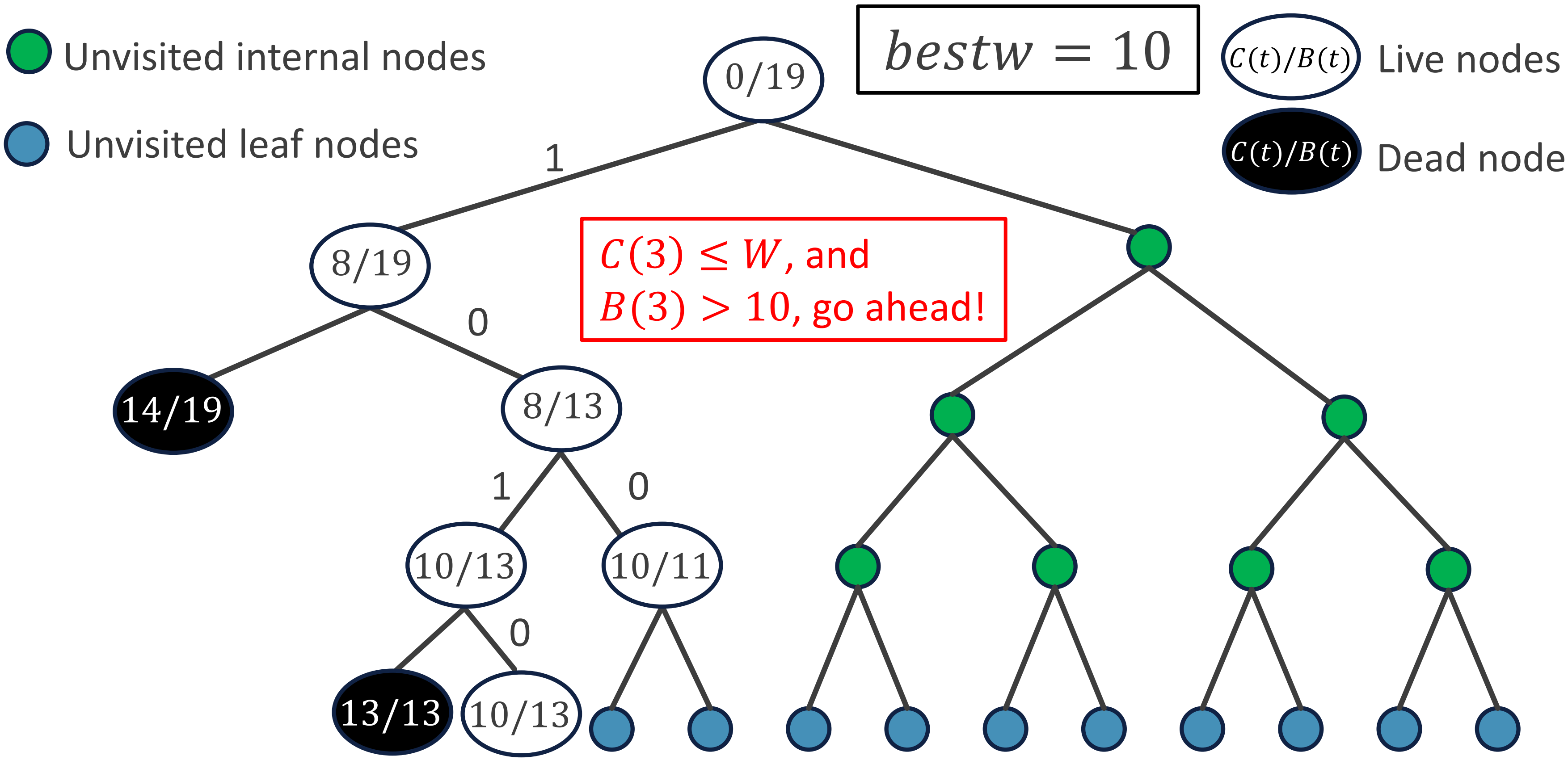
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



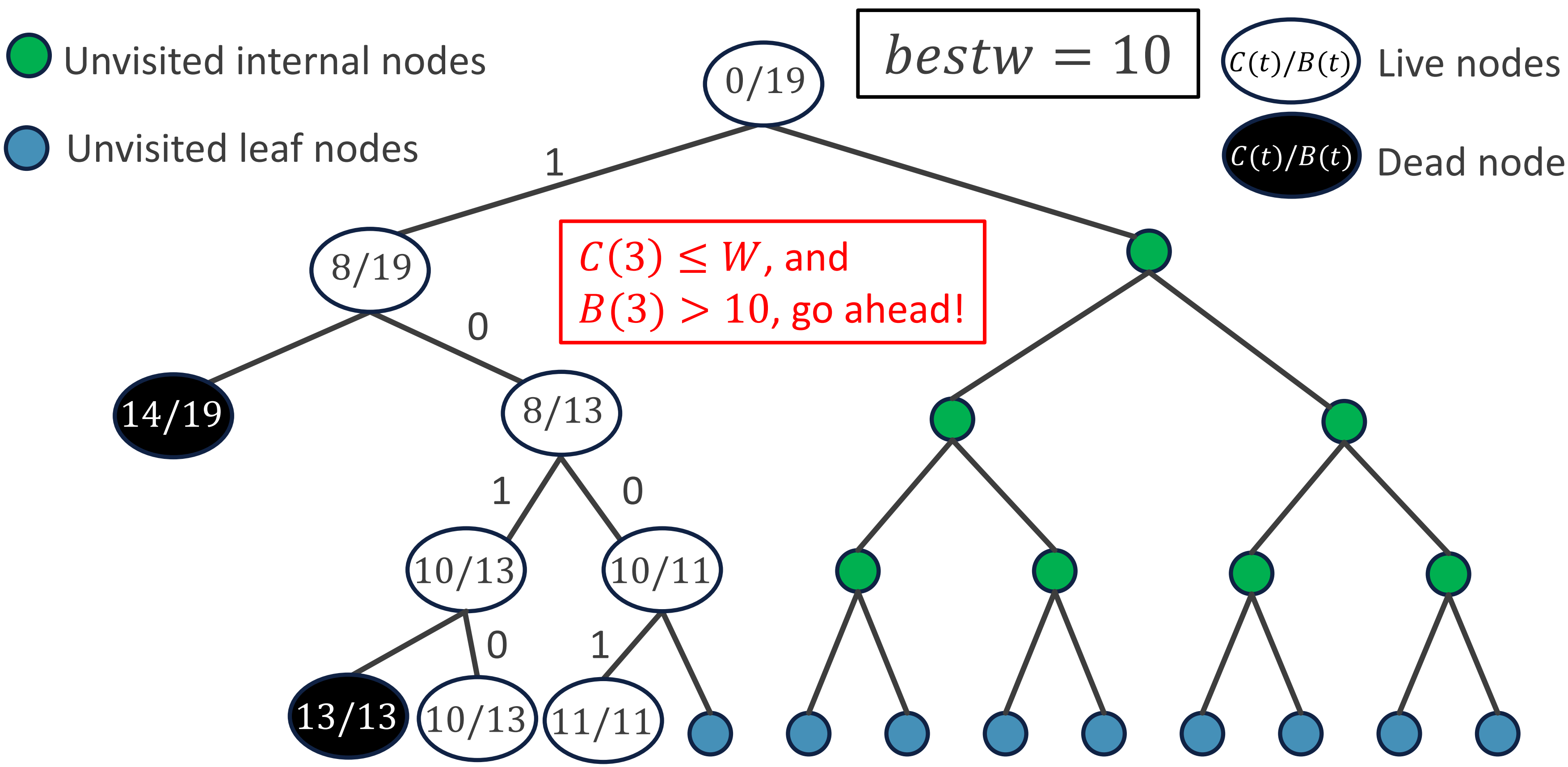
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



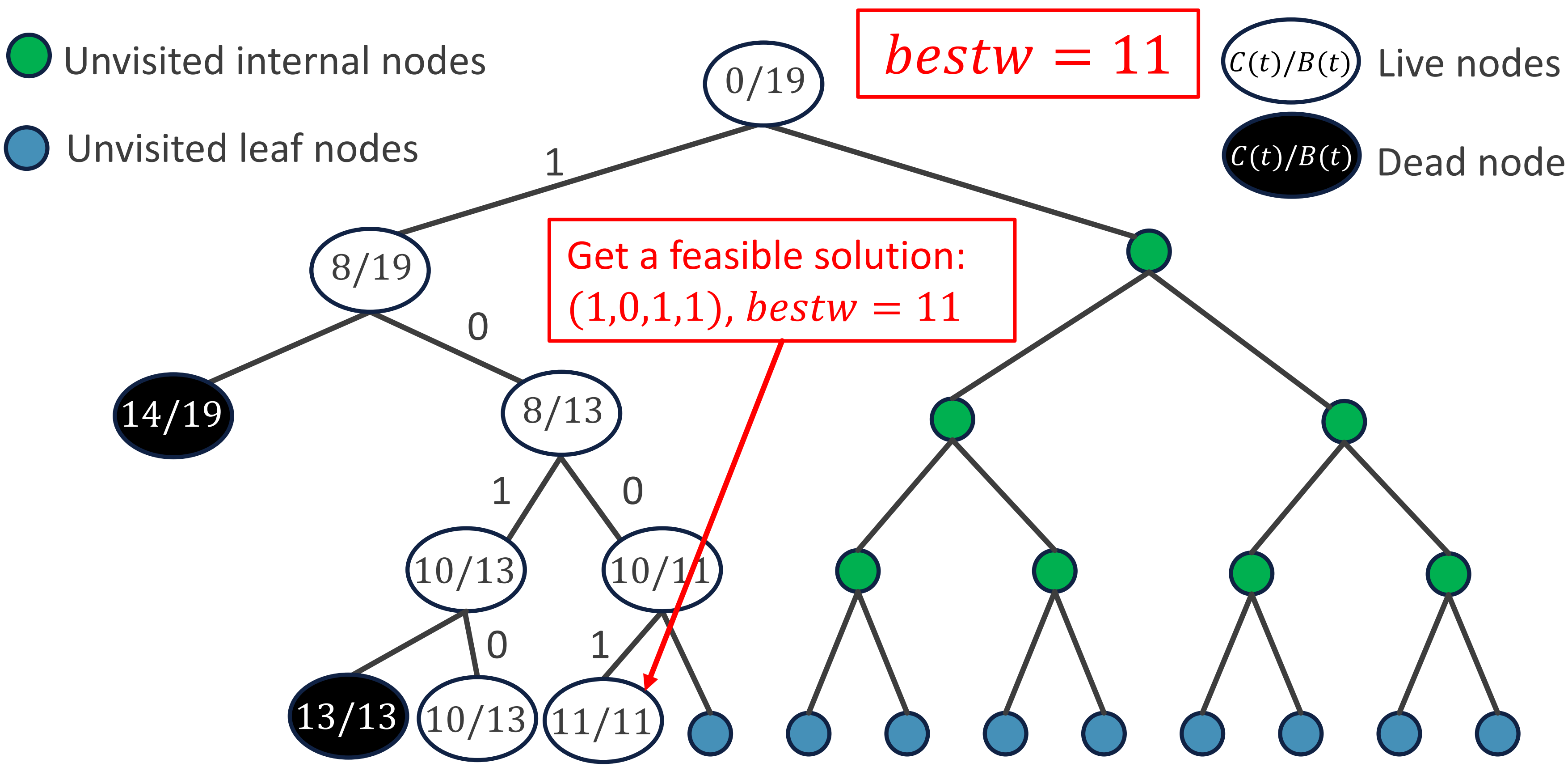
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



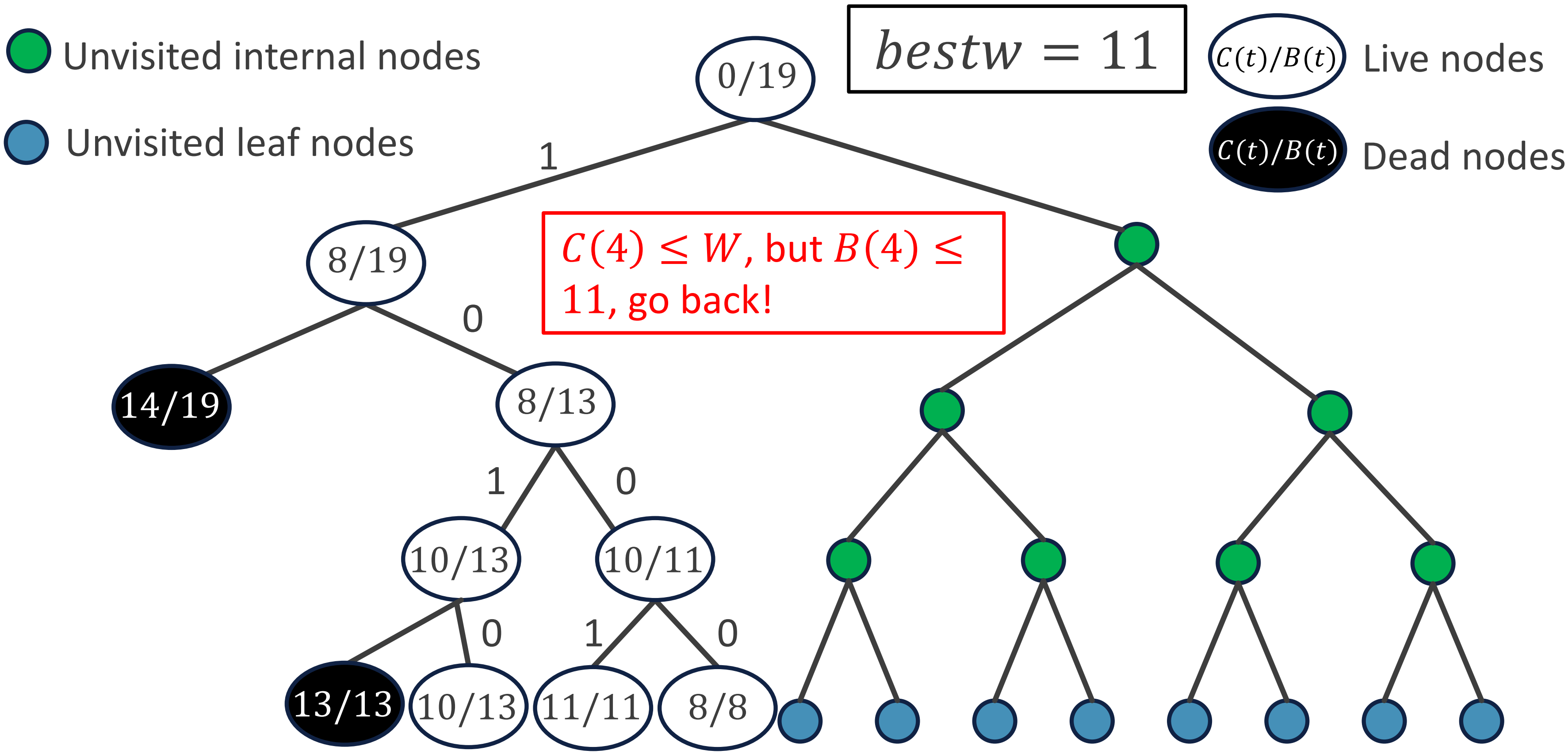
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



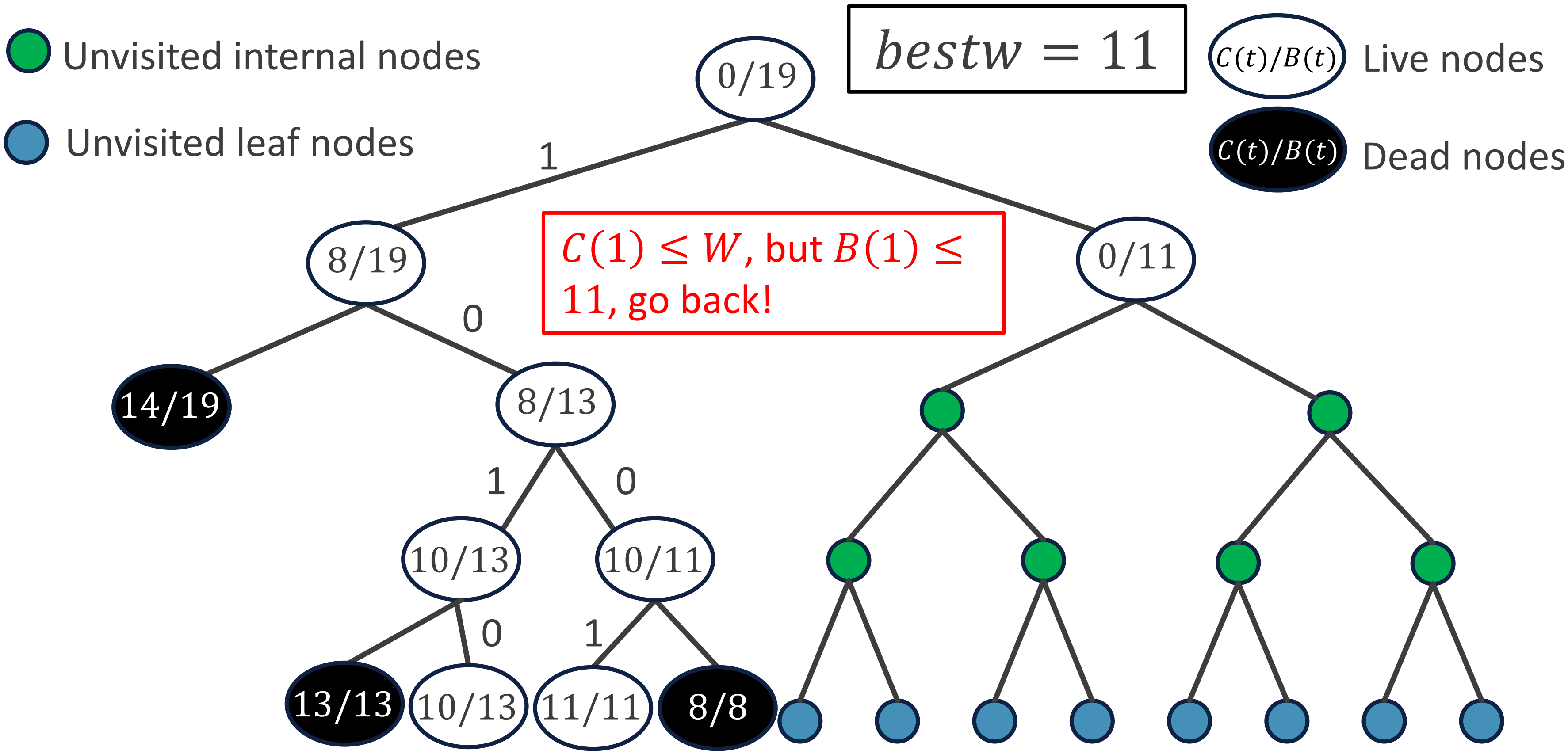
Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example



Backtracking for $n = 4, w = [8,6,2,3], W = 12$

Example



Backtracking for $n = 4$, $w = [8,6,2,3]$, $W = 12$

Example

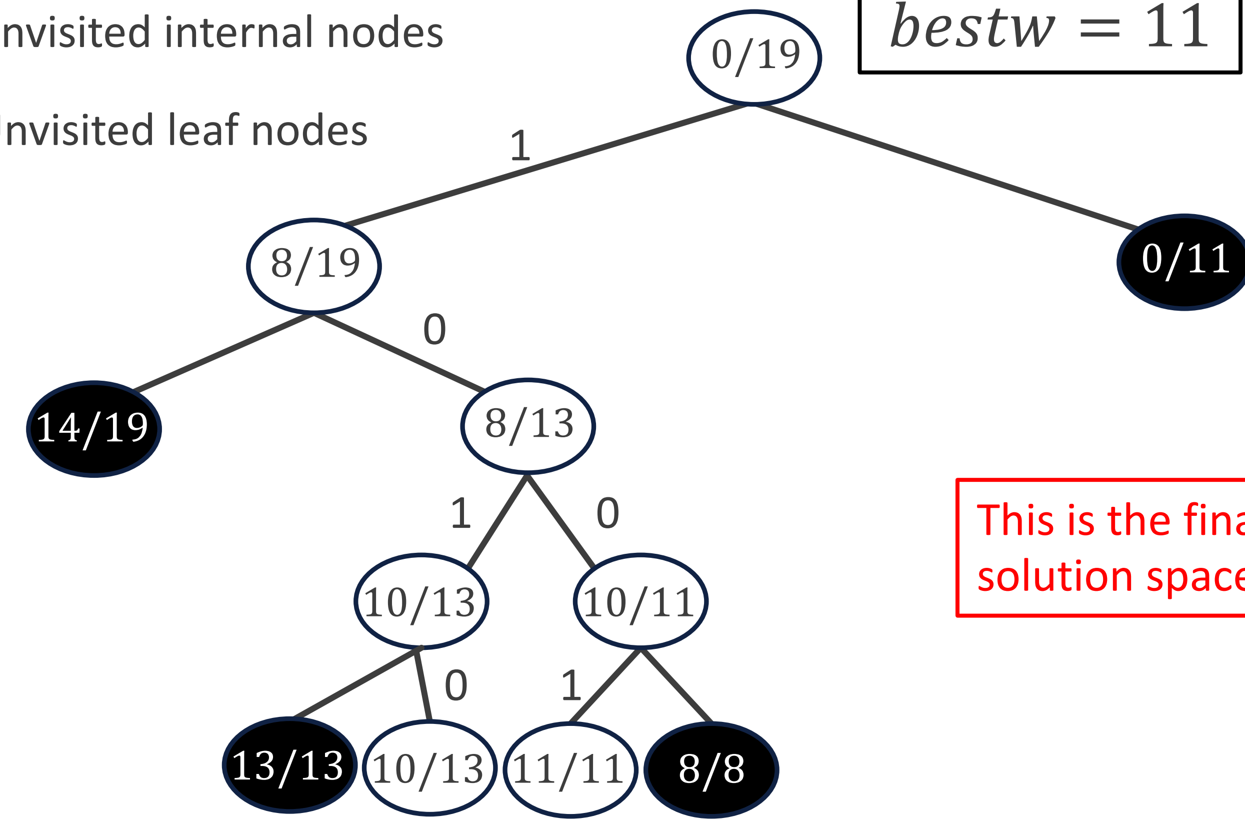
Unvisited internal nodes

Unvisited leaf nodes

$bestw = 11$

$C(t)/B(t)$ Live nodes

$C(t)/B(t)$ Dead nodes



This is the final pruned solution space tree.

Backtracking for $n = 4, w = [8,6,2,3], W = 12$

Pseudocode

```
ImprovedBacktrackLoading(i)
if i > n then
    if cw > bestw then
        bestw ← cw
        for j ← 1 to n do
            bestx[j] ← x[j]
else
    r ← r − w[i]
    if C(i) ≤ W then    x[i] ← 1
                        cw ← cw + w[i]
                        ImprovedBacktrackLoading(i + 1)
                        cw ← cw − w[i]
    if B(i) > bestw then  x[i] ← 0
                        ImprovedBacktrackLoading(i + 1)
    r ← r + w[i]
```

Record the best solution

Record the current solution

r is initialized as the total weight sum and reduced at the begging of each recursive call. After each recursive call, we add the weight back for going back.

Time Complexity

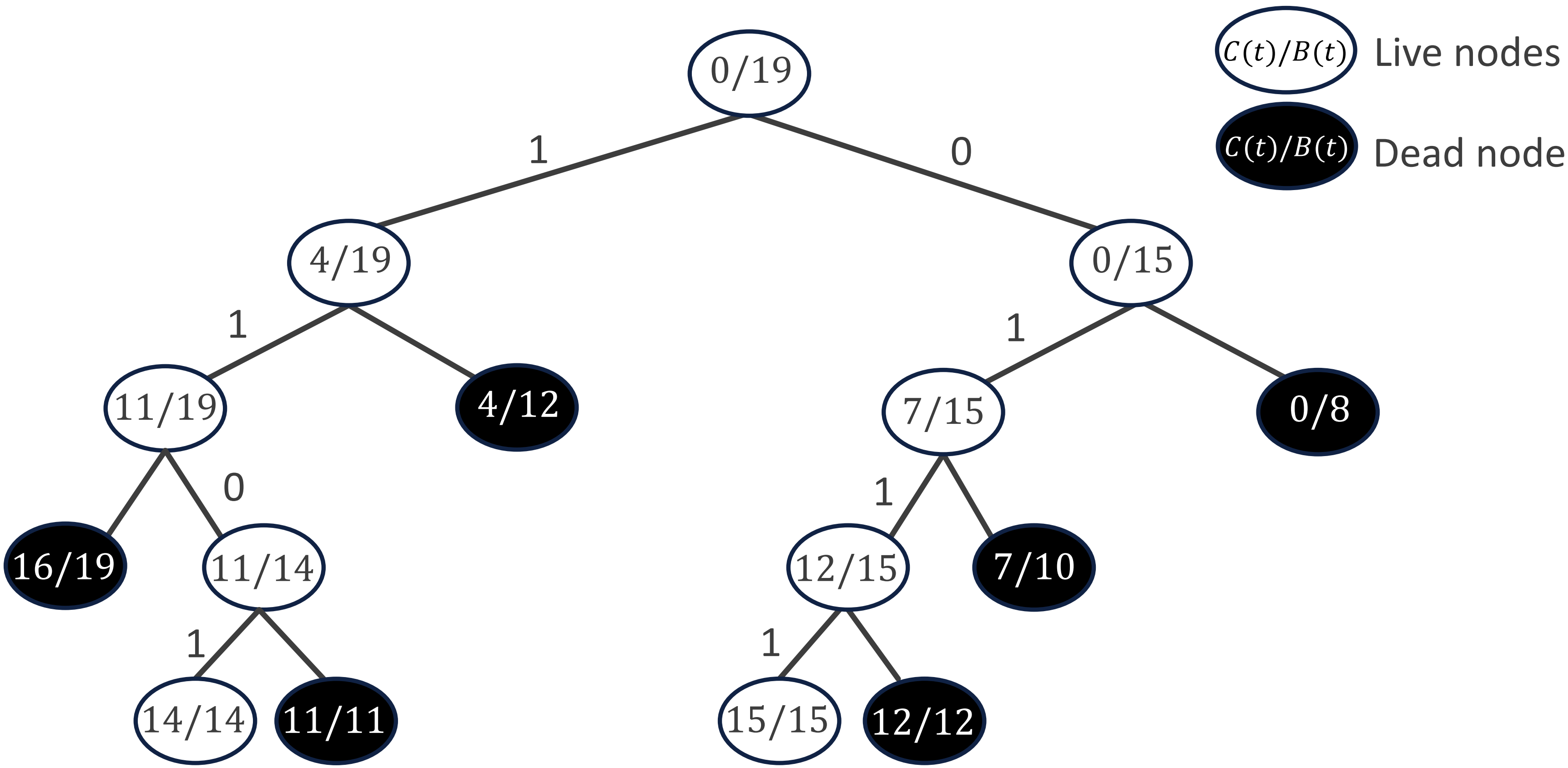
- Although backtracking seems very efficient. The time complexity for this algorithm is $O(n2^n)$.
  - 2^n is the time for searching the solution space.
  - n is the time to store the best solution.
- This is a sad story…

Classroom Exercise

- Draw the pruned solution space tree for the following container loading problem instance.

$$n = 4, w = [4,7,5,3], W = 15$$

Classroom Exercise



Backtracking for $n = 4, w = [4,7,5,3], W = 15$

Classroom Exercise

- In the Sum-of-Subsets problem, there are n positive integers (weights) w_i and a positive integer W.
- The goal is to find all subsets of the integers that sum to W.
- Example:
  - Suppose that $n = 4$, $W = 13$, and $w = [3,4,5,6]$.
  - The solutions is $[1,1,0,1]$ because $w_1 + w_2 + w_4 = 3 + 4 + 6 = 13$,
- Design the constraint function and bounding function, and the corresponding condition.
- Draw the pruned solution space tree of the above example.

Classroom Exercise

- The constraint function $C(i)$ and its condition are same as the container loading problem:

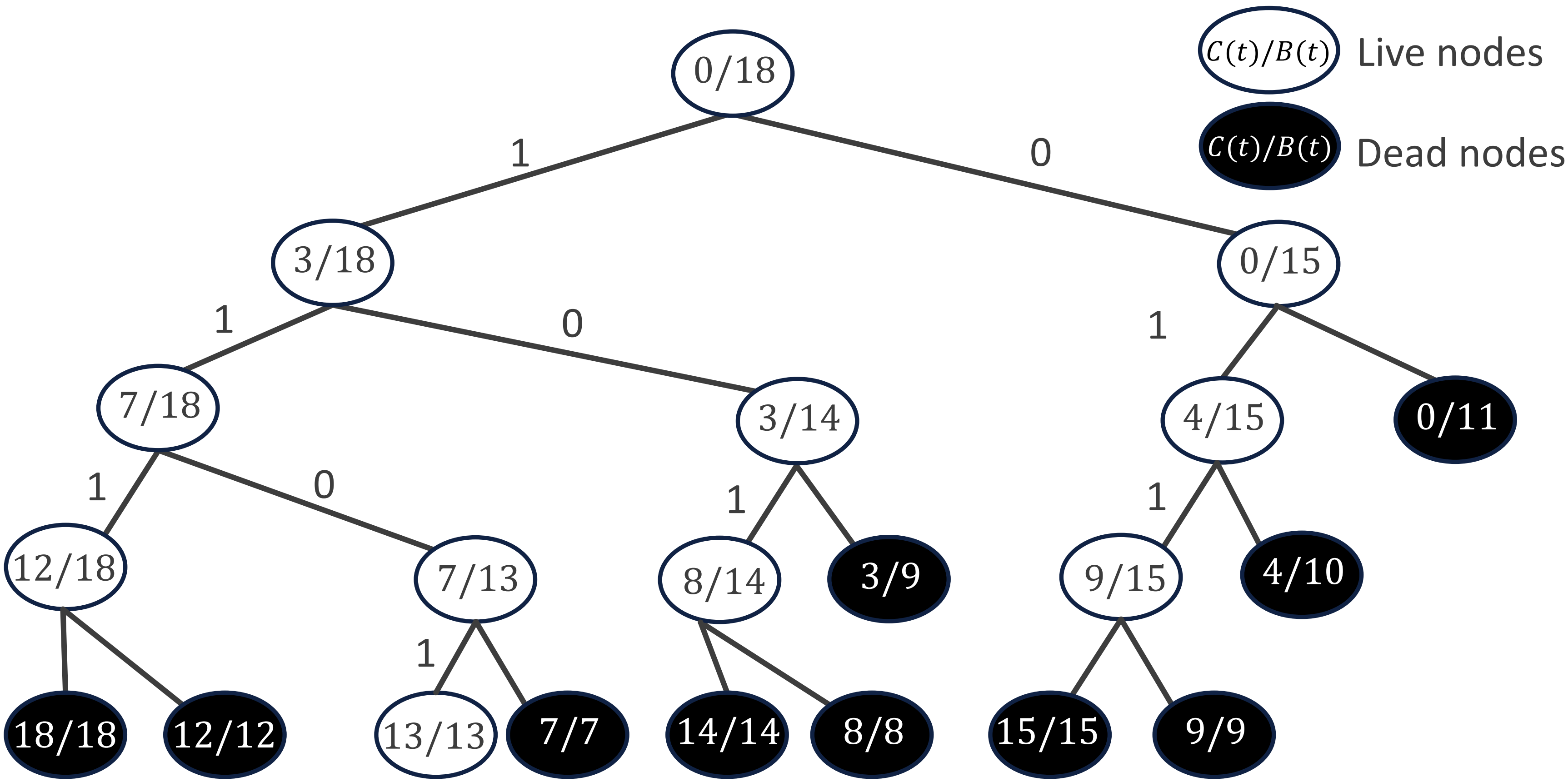
$$C(i) > W$$

- The bounding function $B(i)$ is same as the container loading problem, but the condition is different:

$$B(i) < W$$

  - Instead of comparing with $bestw$ in the container loading problem.

Classroom Exercise



Backtracking for $n = 4$, $w = [3,4,5,6]$, $W = 13$

0/1 KNAPSACK PROBLEM

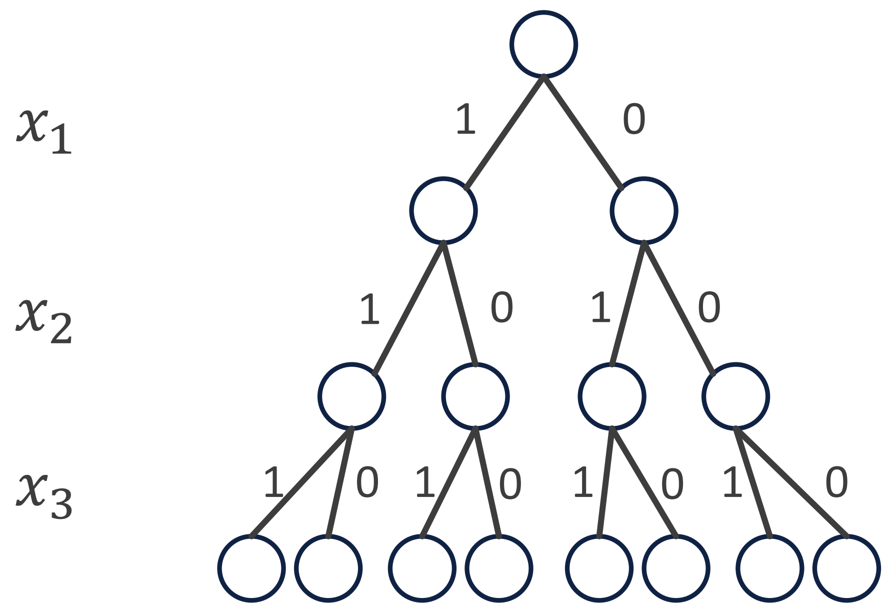
0/1 Knapsack Problem

- There are n items: the ith item is worth v_i dollars and weights w_i kg. The capacity of knapsack is W kg.
- Assuming that the solutions are represented by vectors $(x_1, x_2, \dots, x_n)$, where $x_i \in \{0,1\}$. 1 denotes taking item i and 0 denotes not taking item i.
- The 0/1 knapsack problem can be formally stated as follows:

$$\max \sum_{i=1}^{n} v_i x_i \qquad s.t. \sum_{i=1}^{n} w_i x_i \le W$$

0/1 Knapsack Problem

- It is nothing but a high-level container loading problem.
- The size of solution space and the solution space tree are exactly same as the container loading problem.



Solution space tree with $n = 3$

0/1 Knapsack Problem

- Constraint function: also exactly same as the container loading problem!
- Let $cw(i)$ denote the current weight up to level i, namely

$$cw(i) = \sum_{j=1}^{i} w_j x_j$$

then the constraint function is

$$C(i) = cw(i-1) + w_i$$

- The pruning condition is $C(i) > W$, which means there is no capacity to take container i.

0/1 Knapsack Problem

- The bounding function:

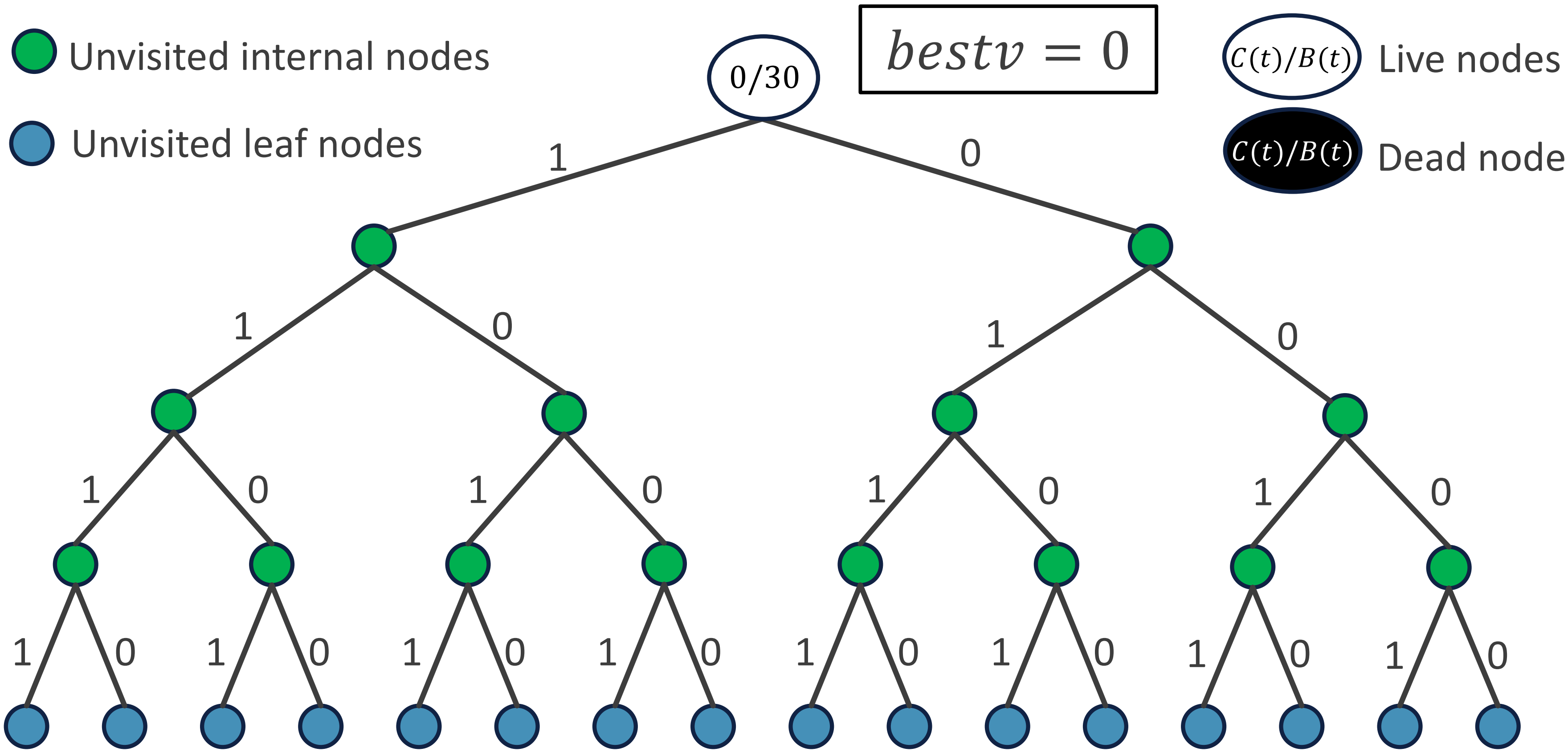
$$B(i) = C(i) + r(i)$$

However, different from the bounding function in the container loading problem, $r(i)$ denotes the value sum of the remaining items, namely,

$$r(i) = \sum_{j=i+1}^{n} v_j$$

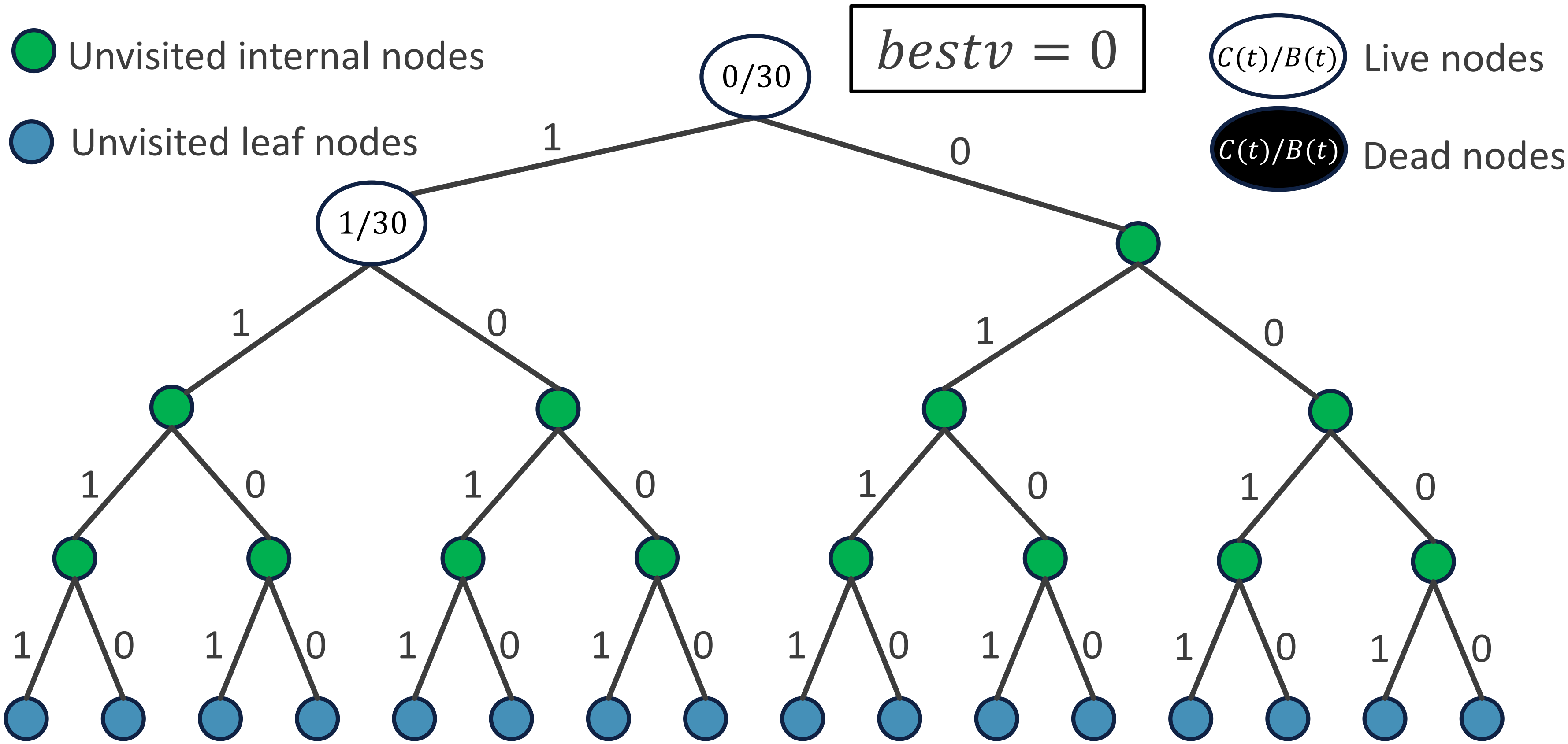
- The pruning condition is $B(i) \leq bestv$, which means the continuing searching along this branch will not give better solution.

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



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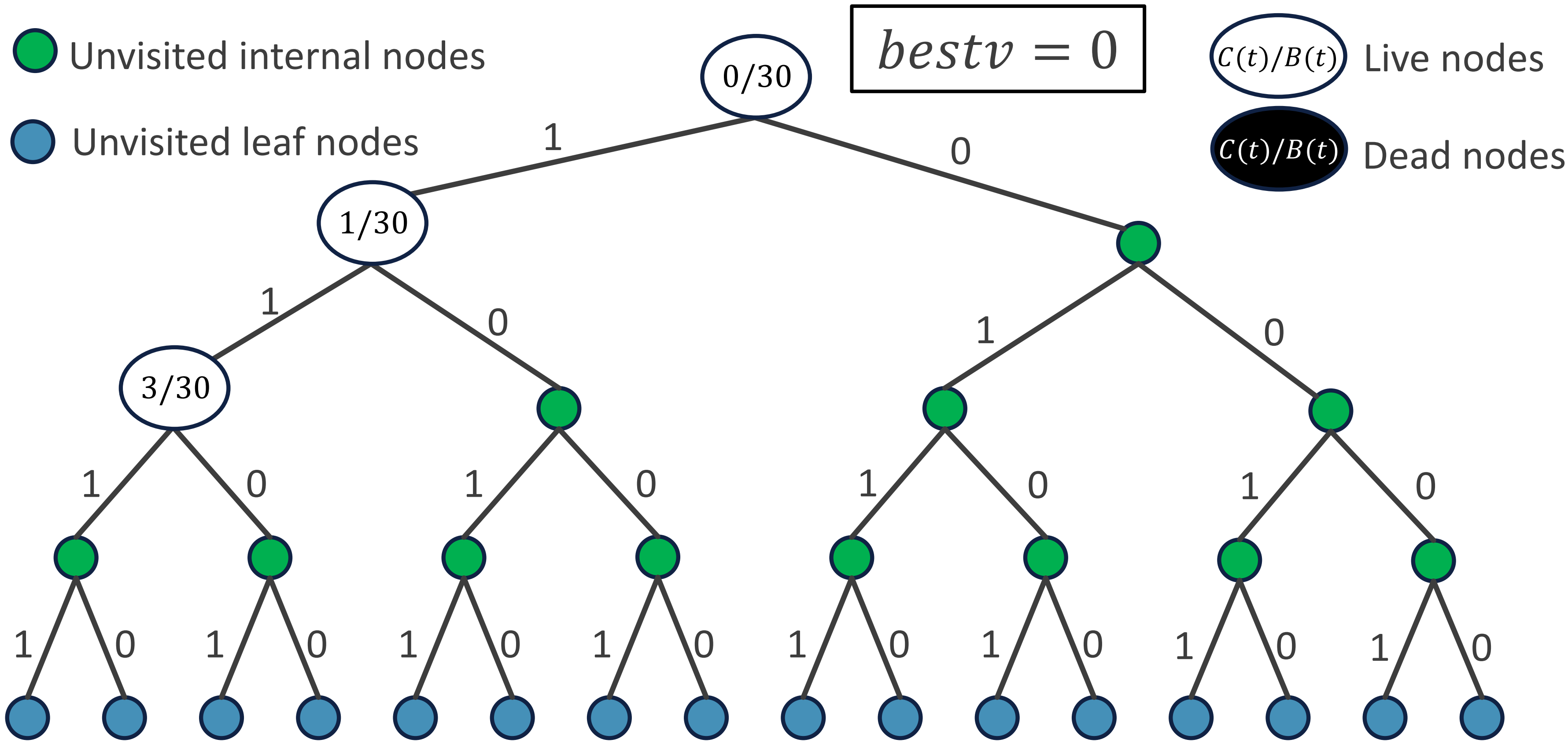
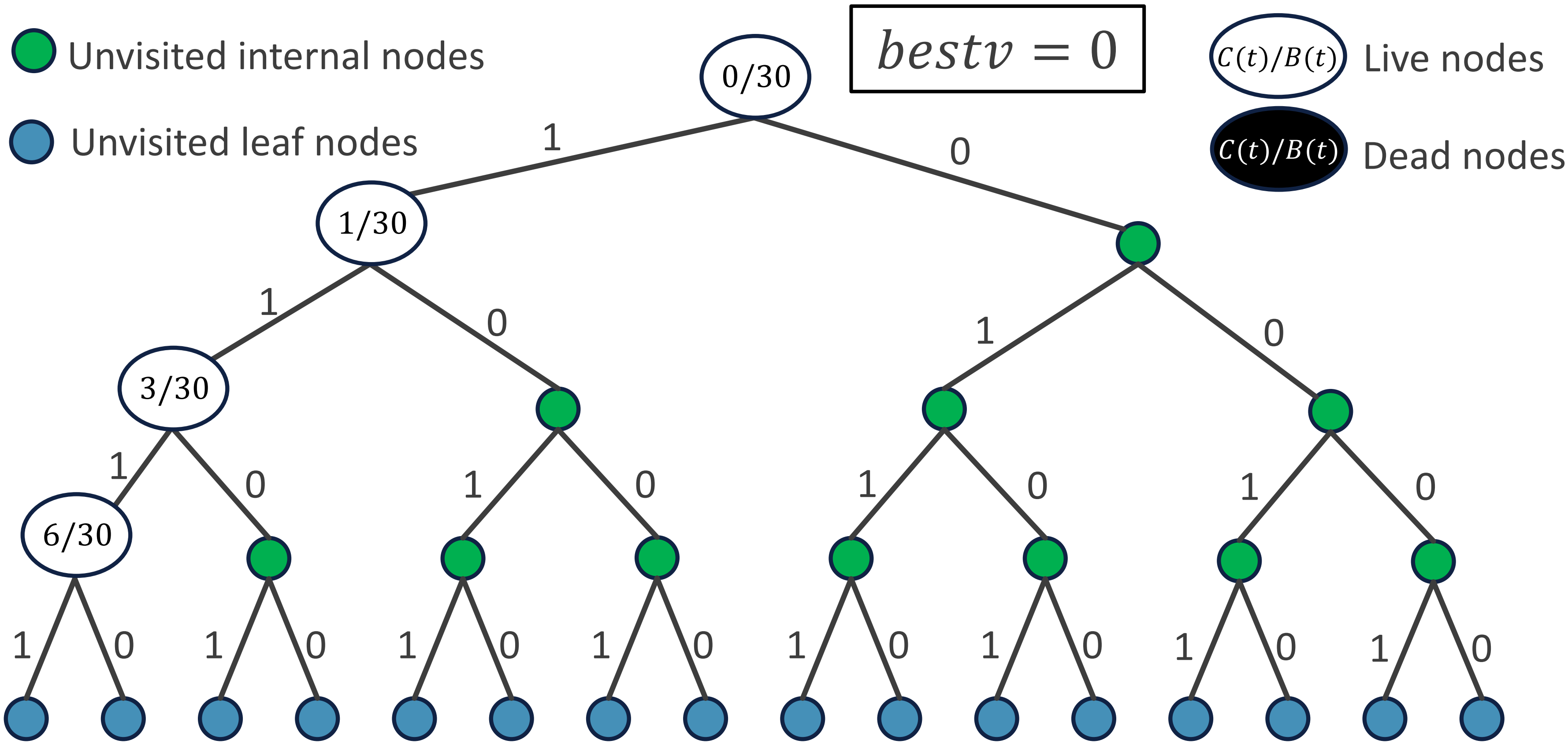
Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

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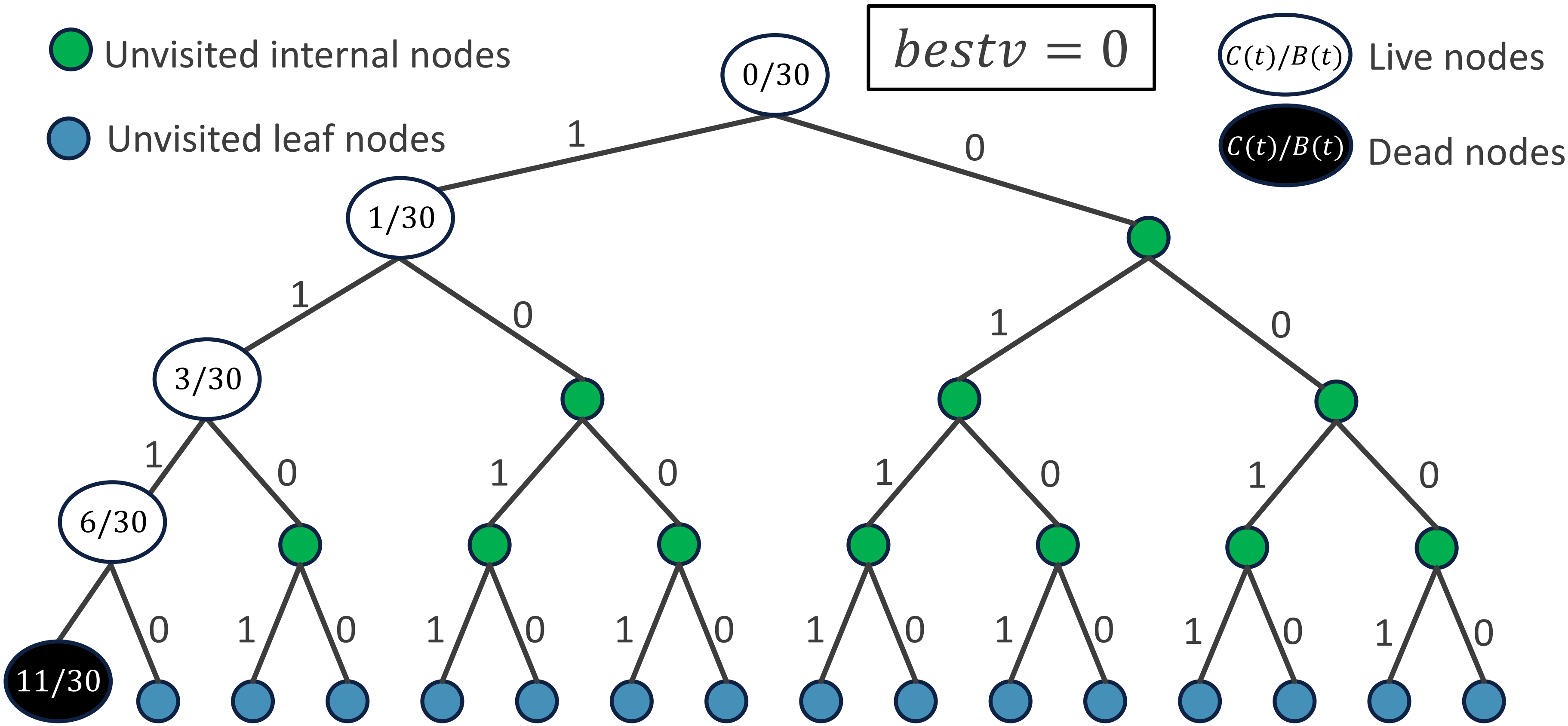
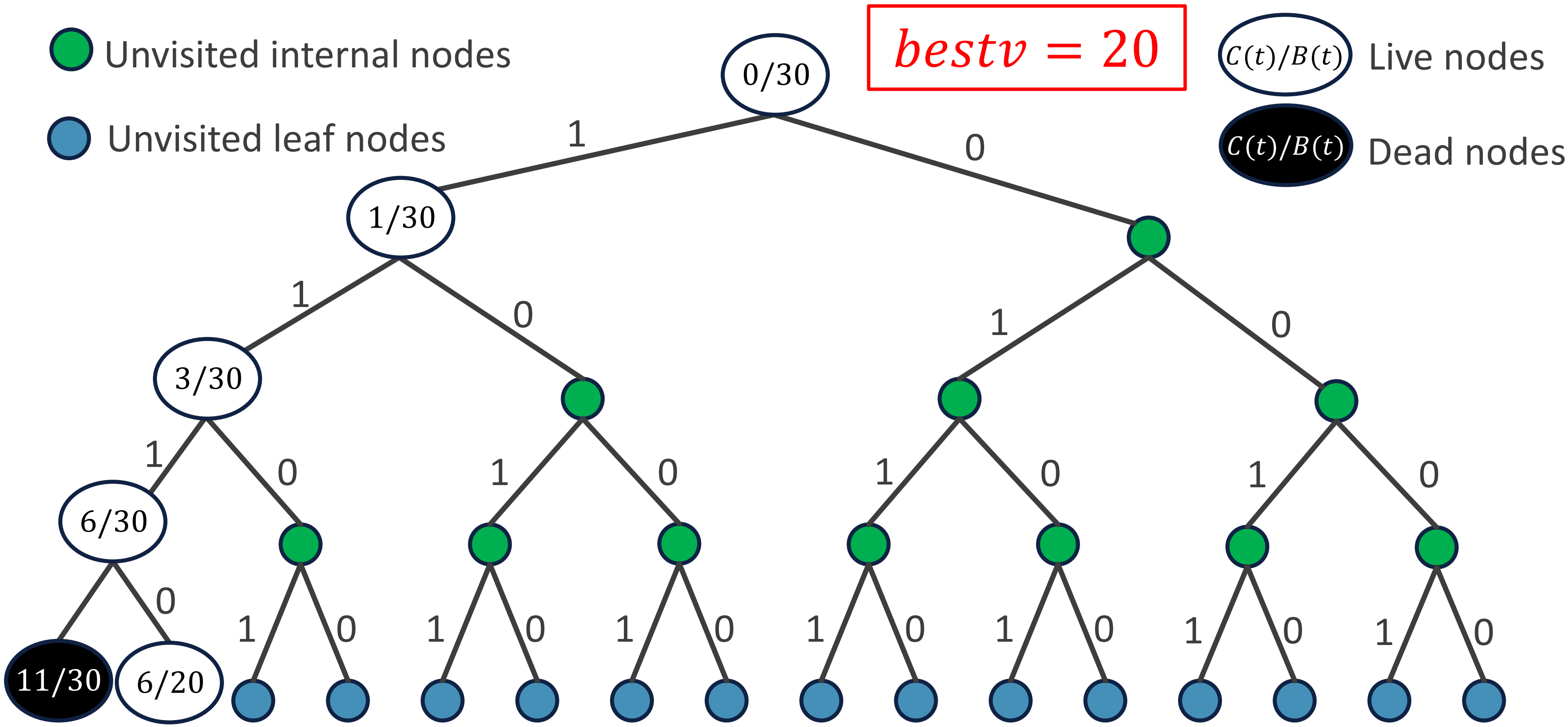
Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



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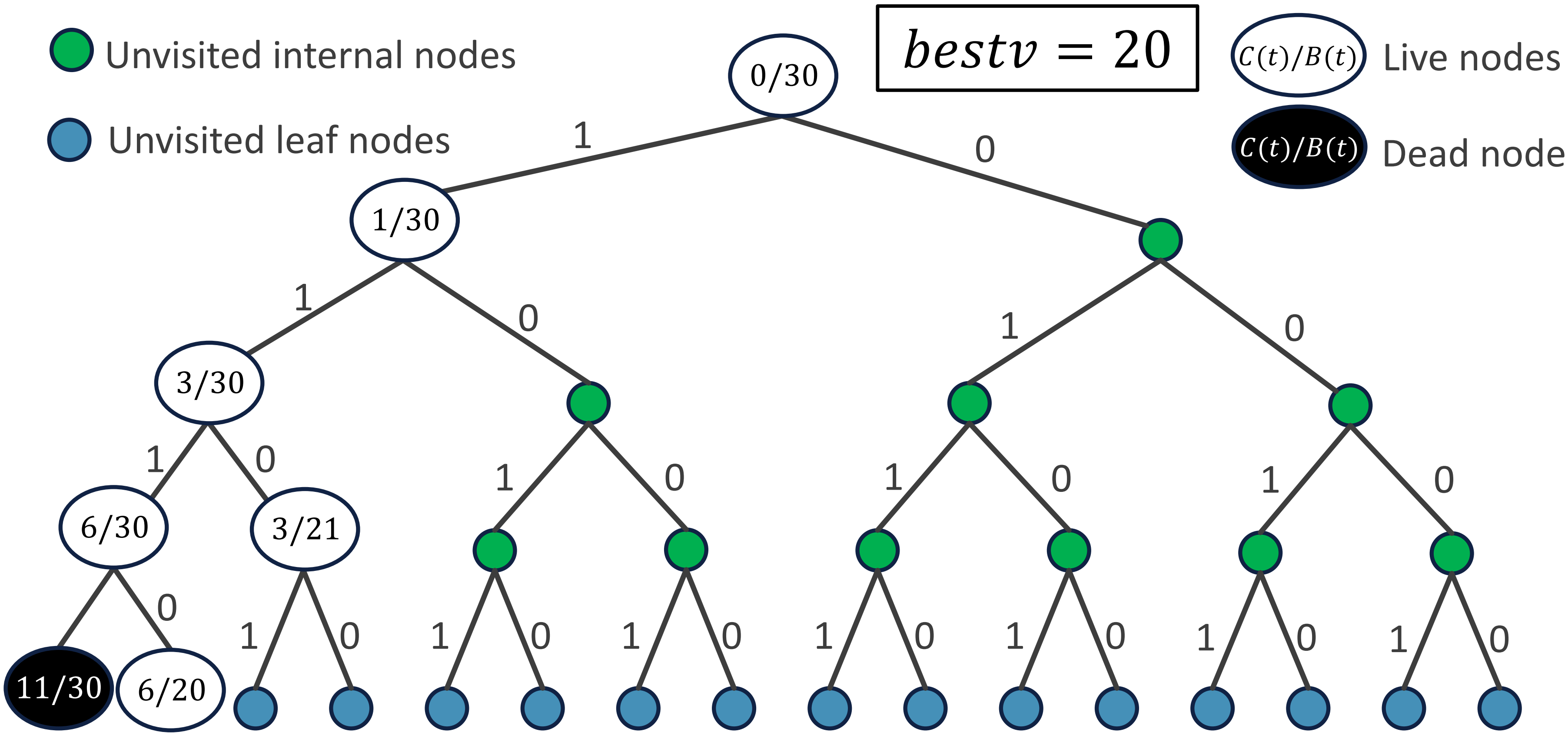
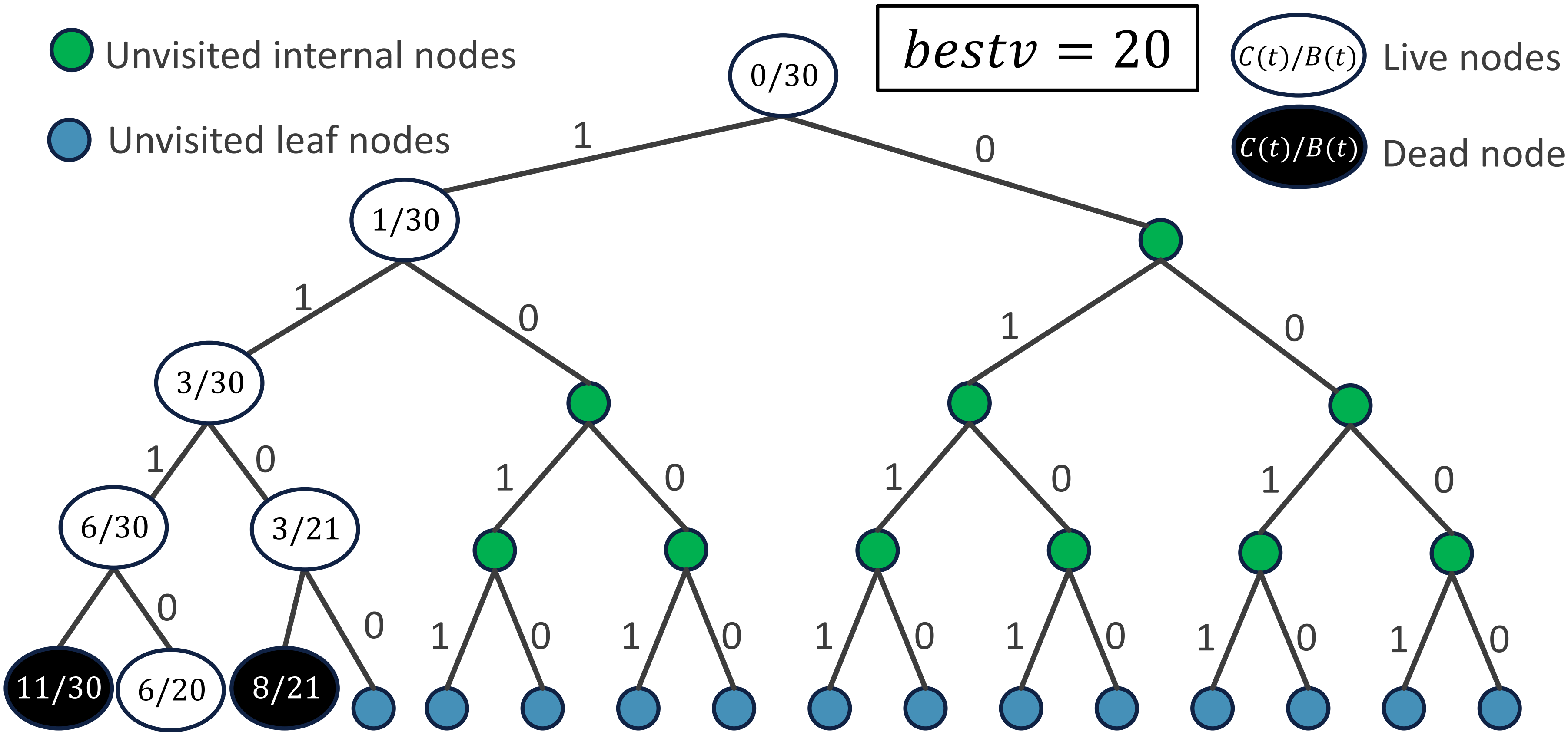
Example



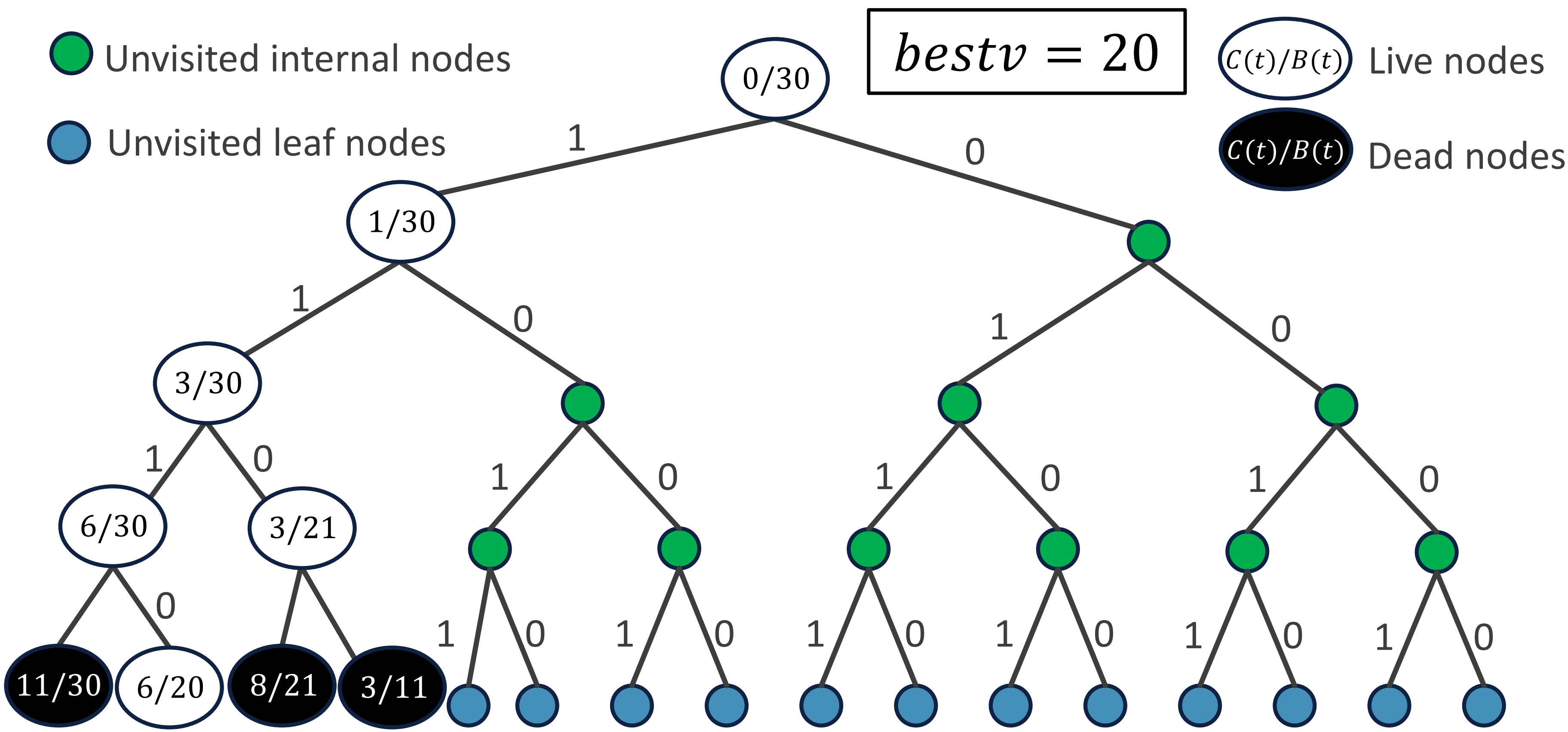
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

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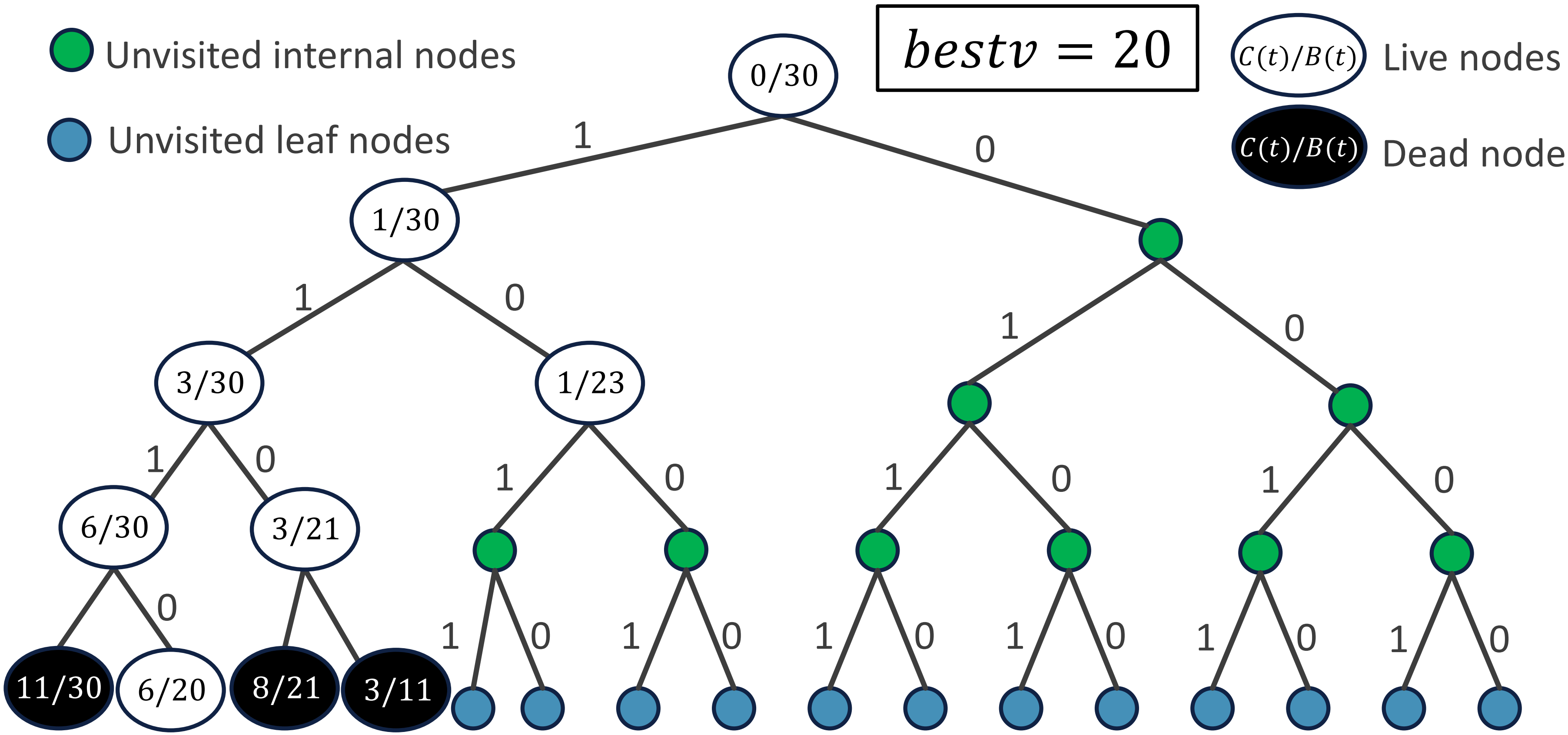

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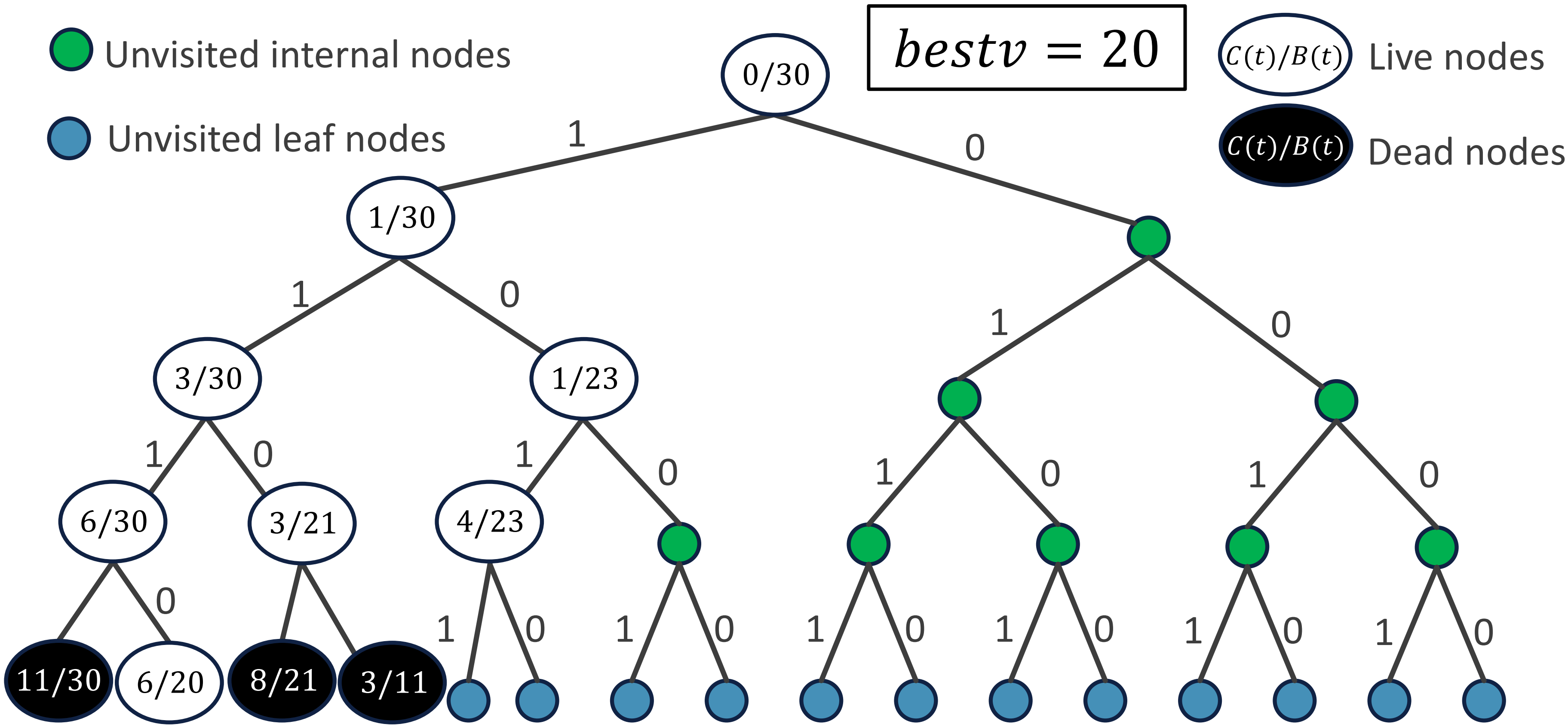
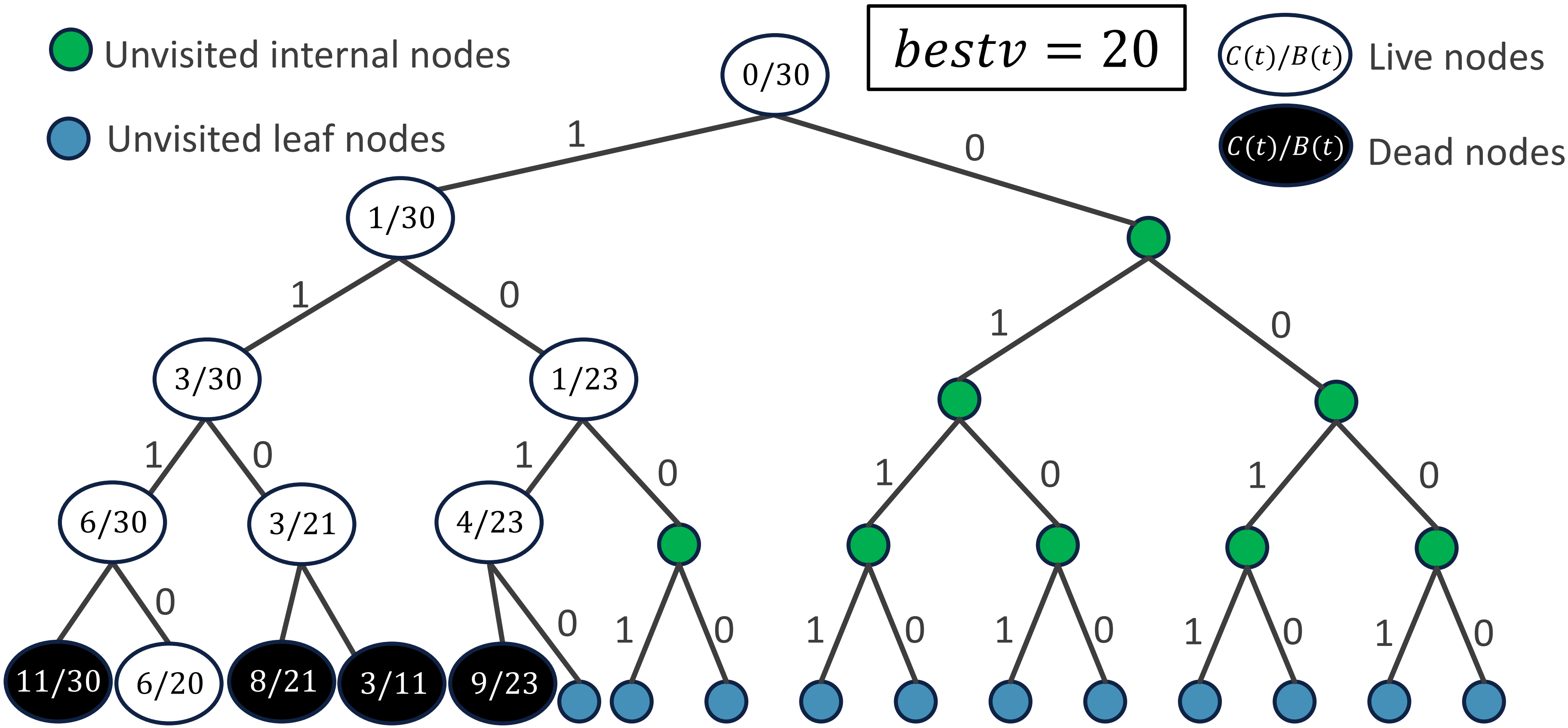
Example



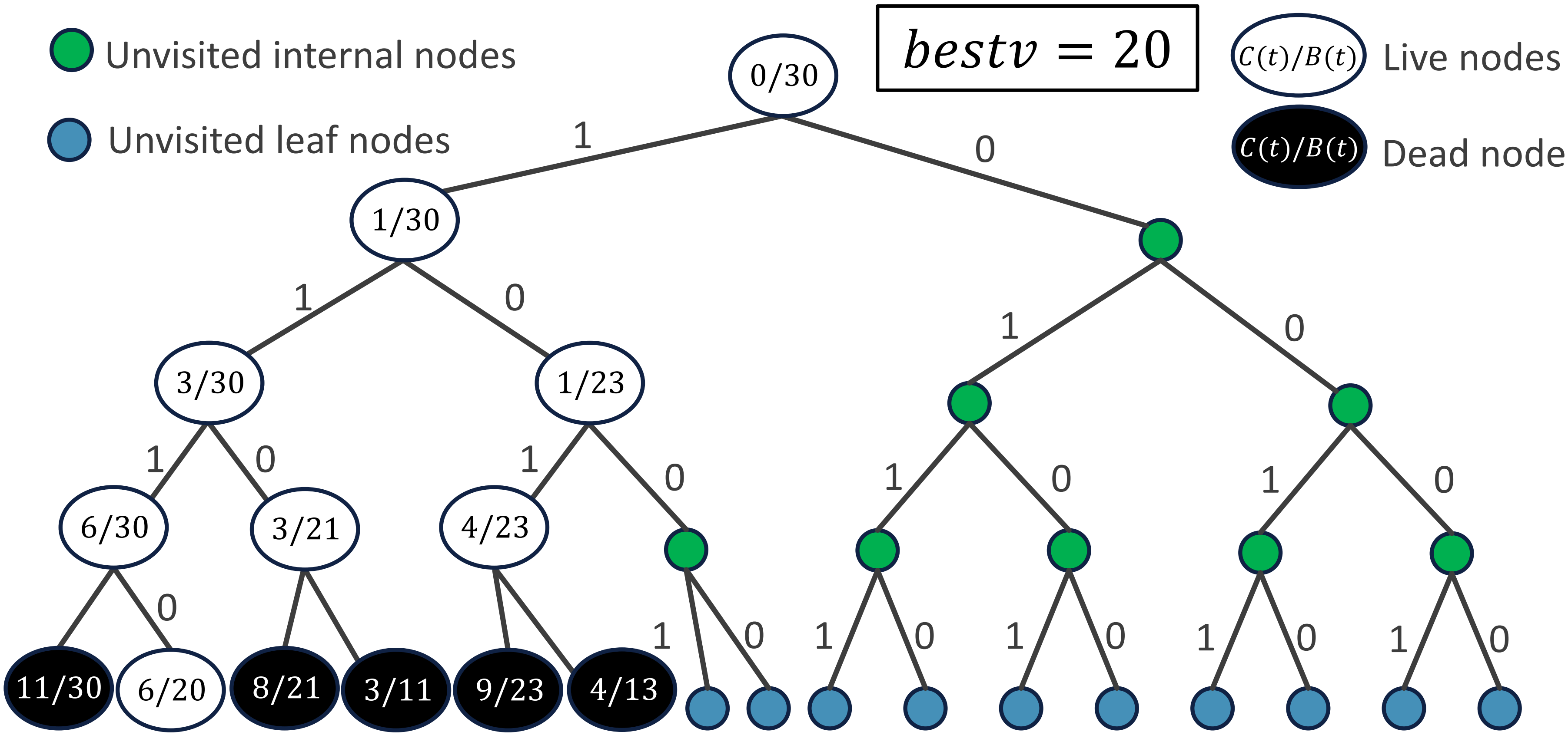
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



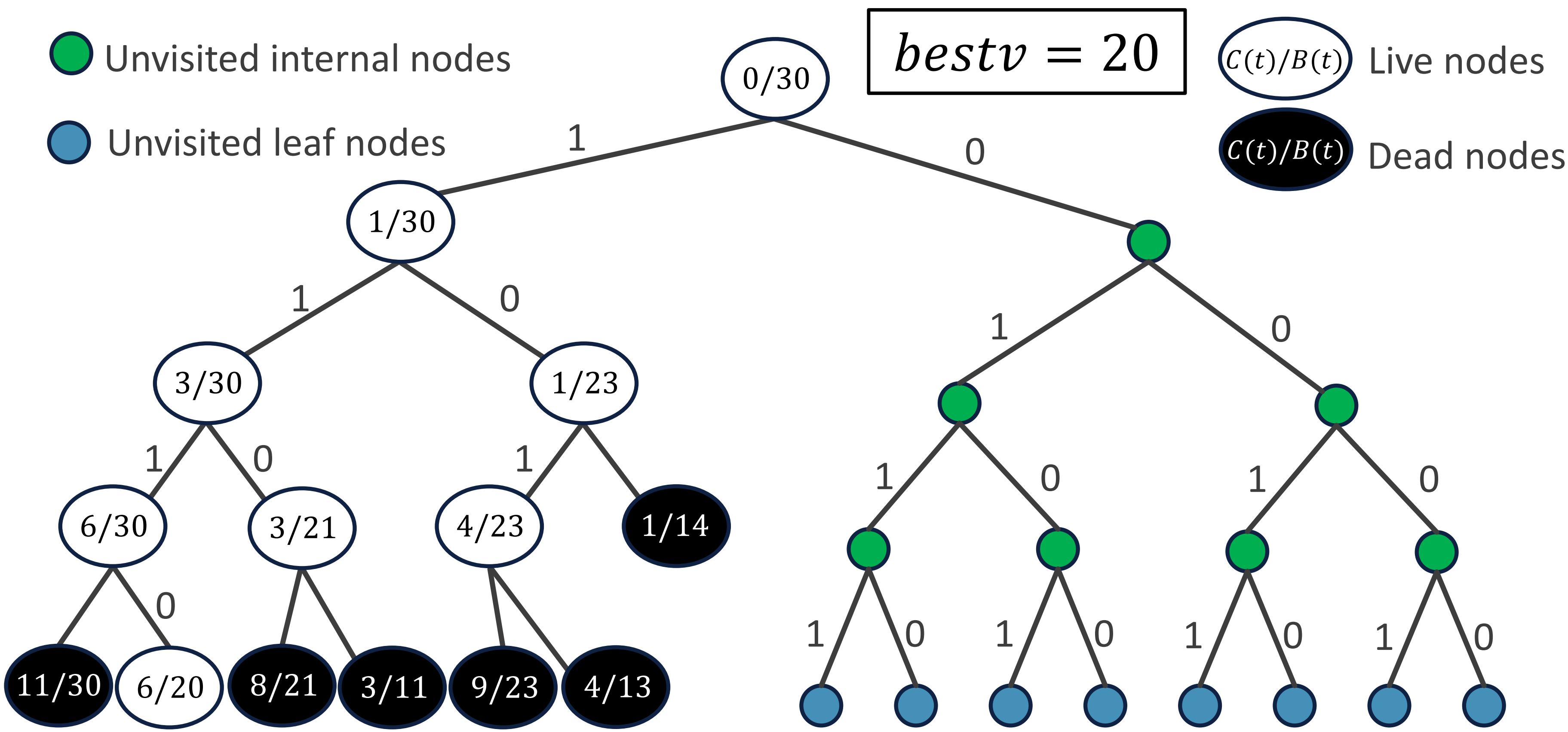
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



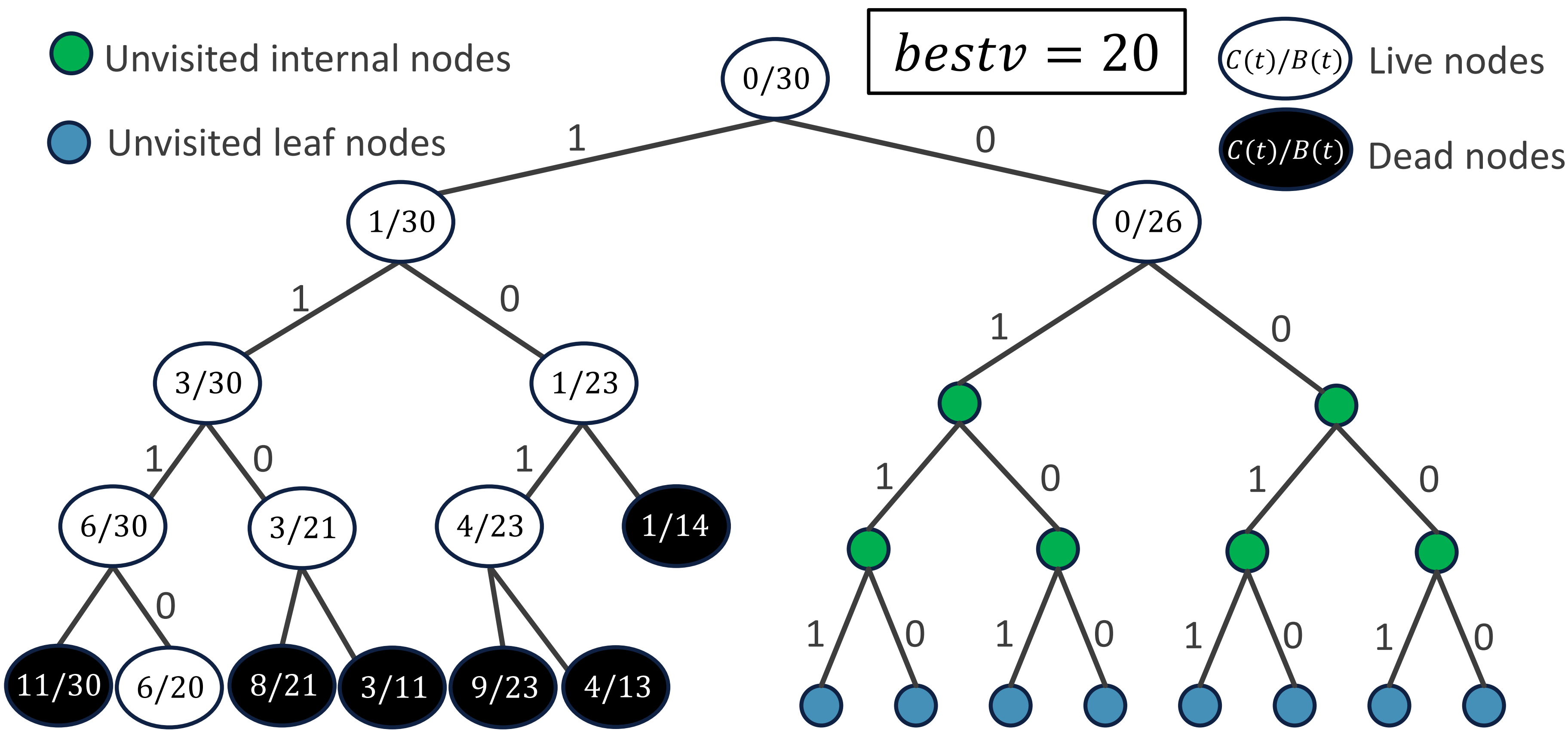
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



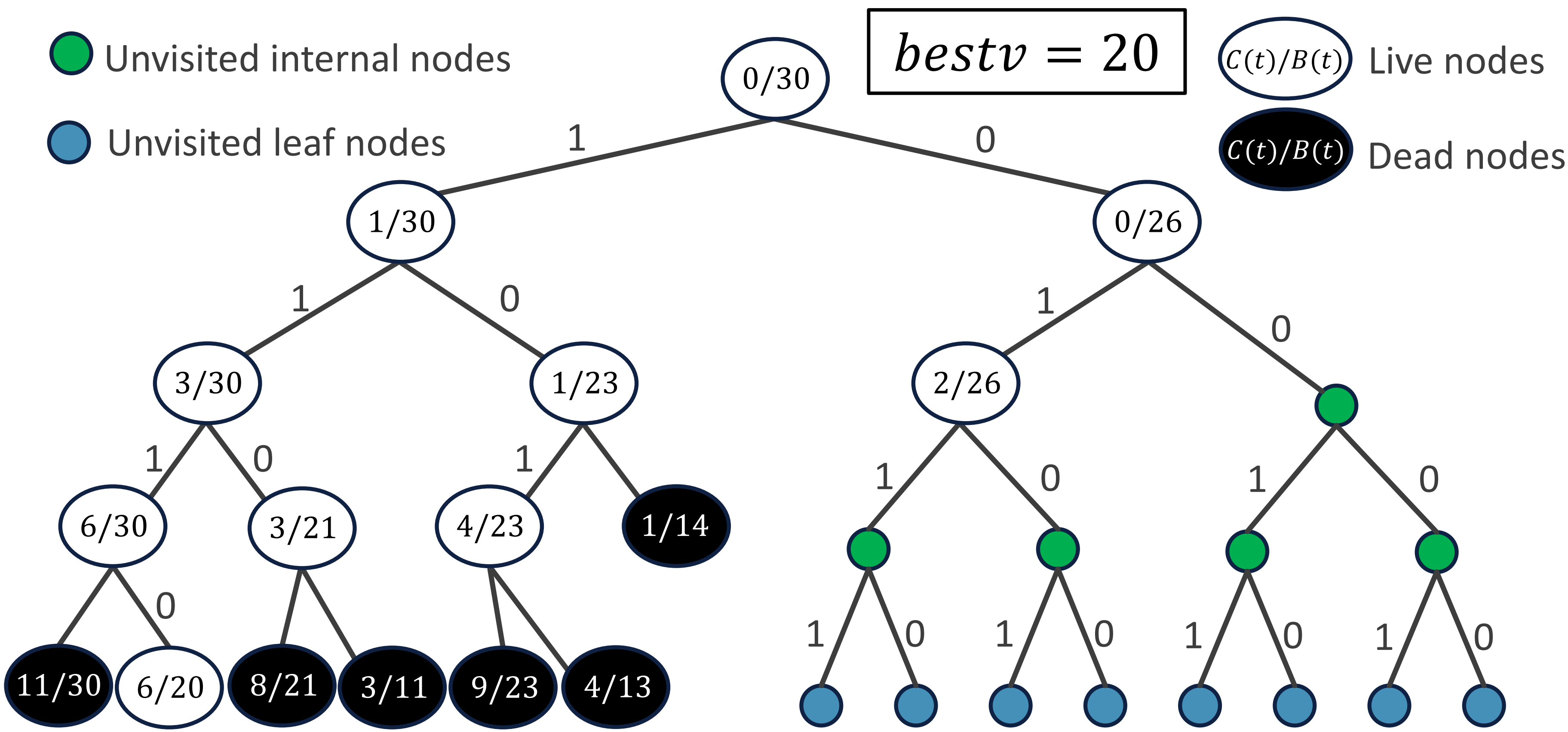
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



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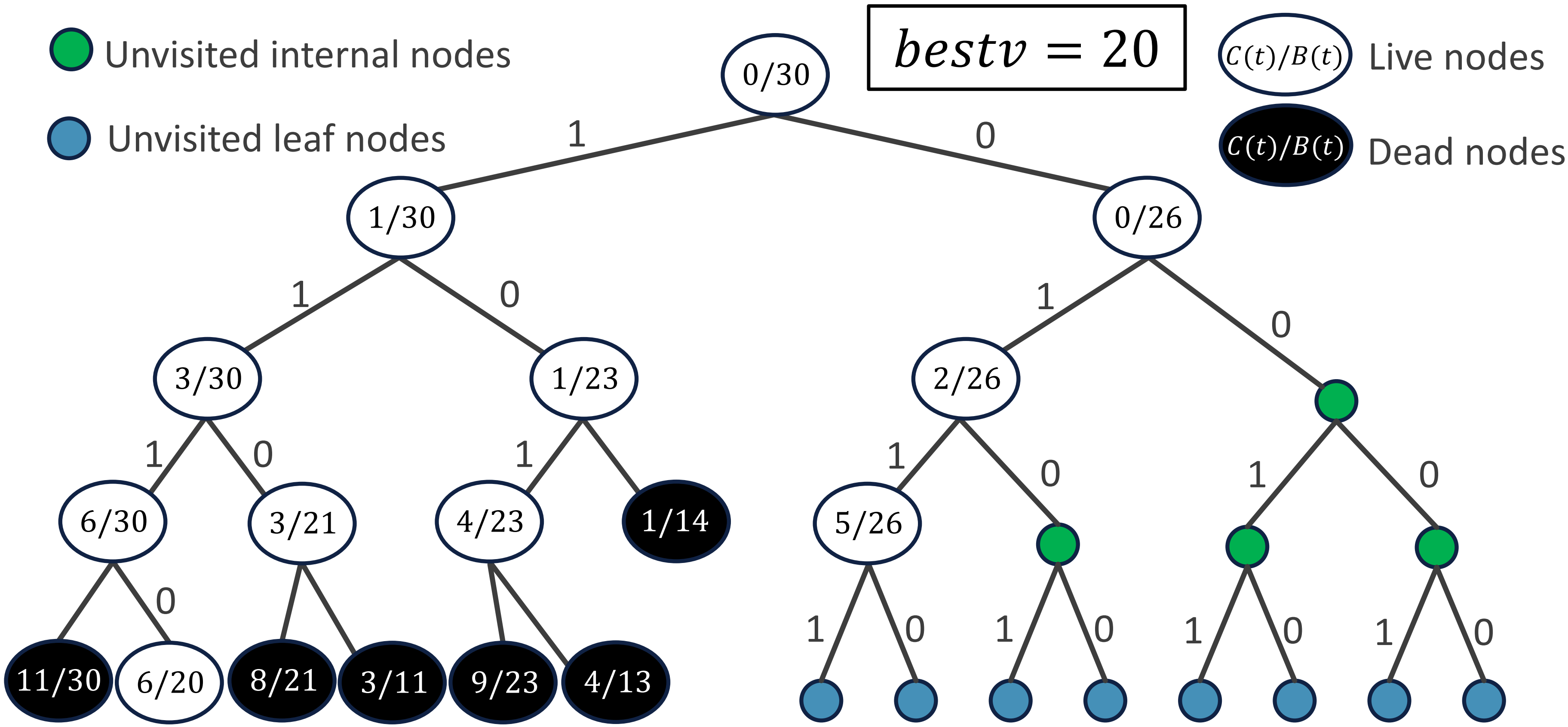

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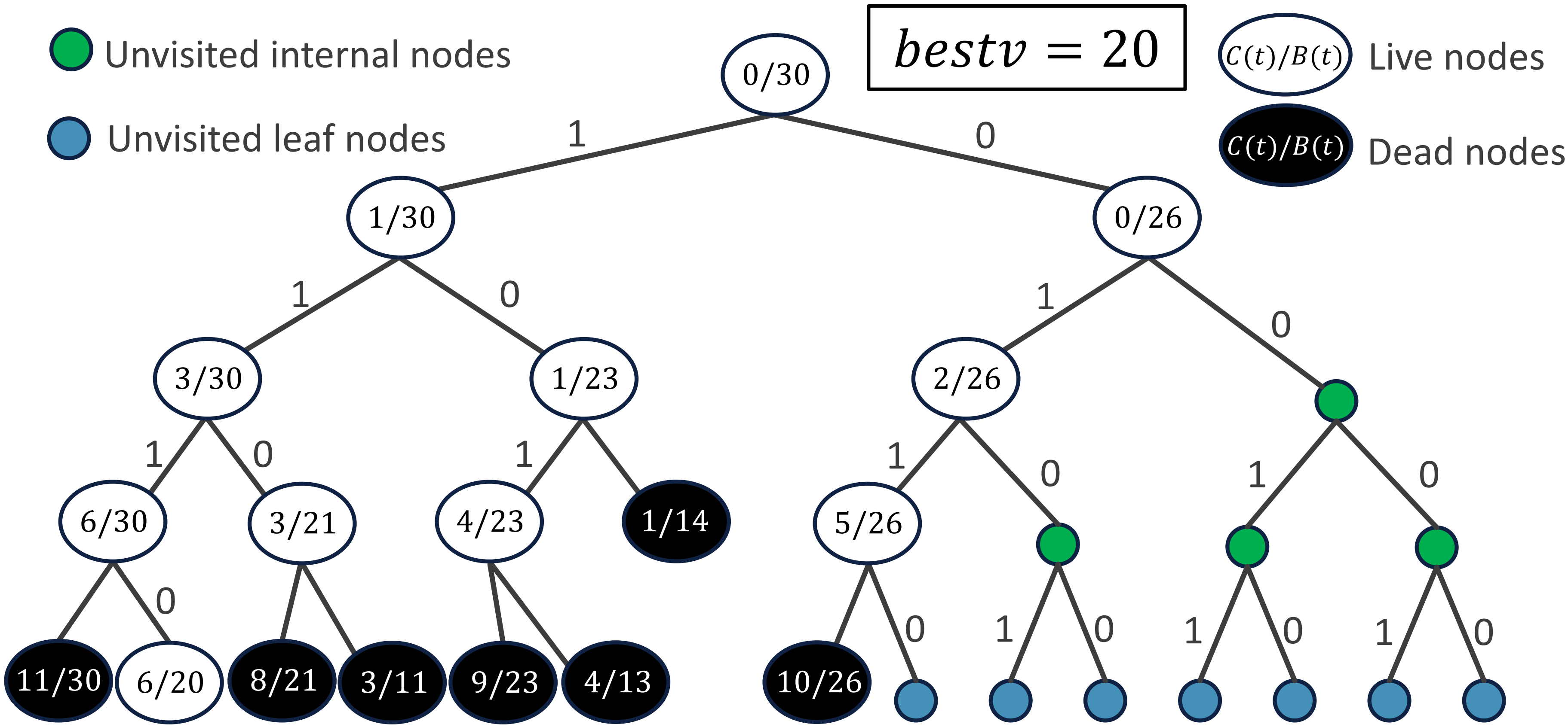
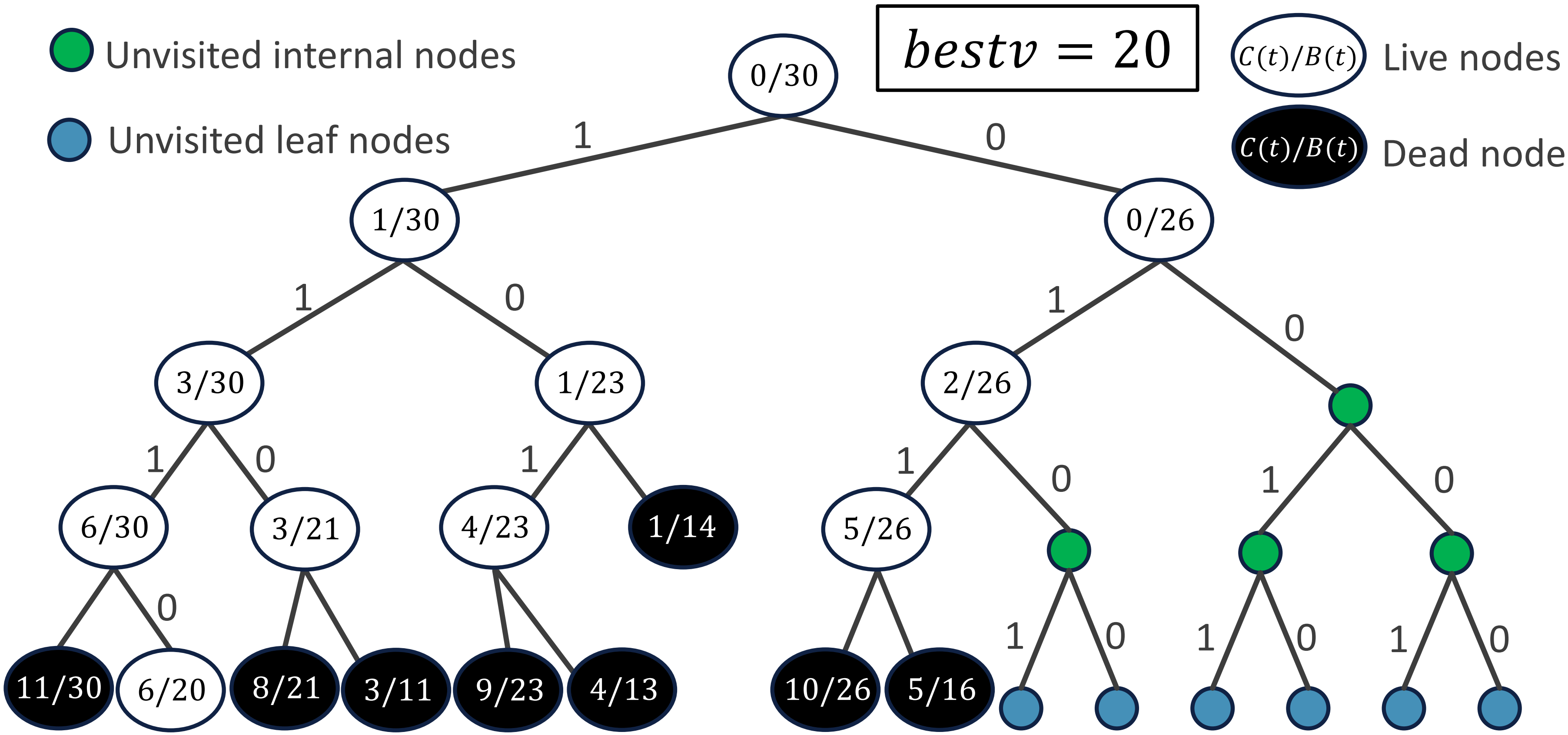


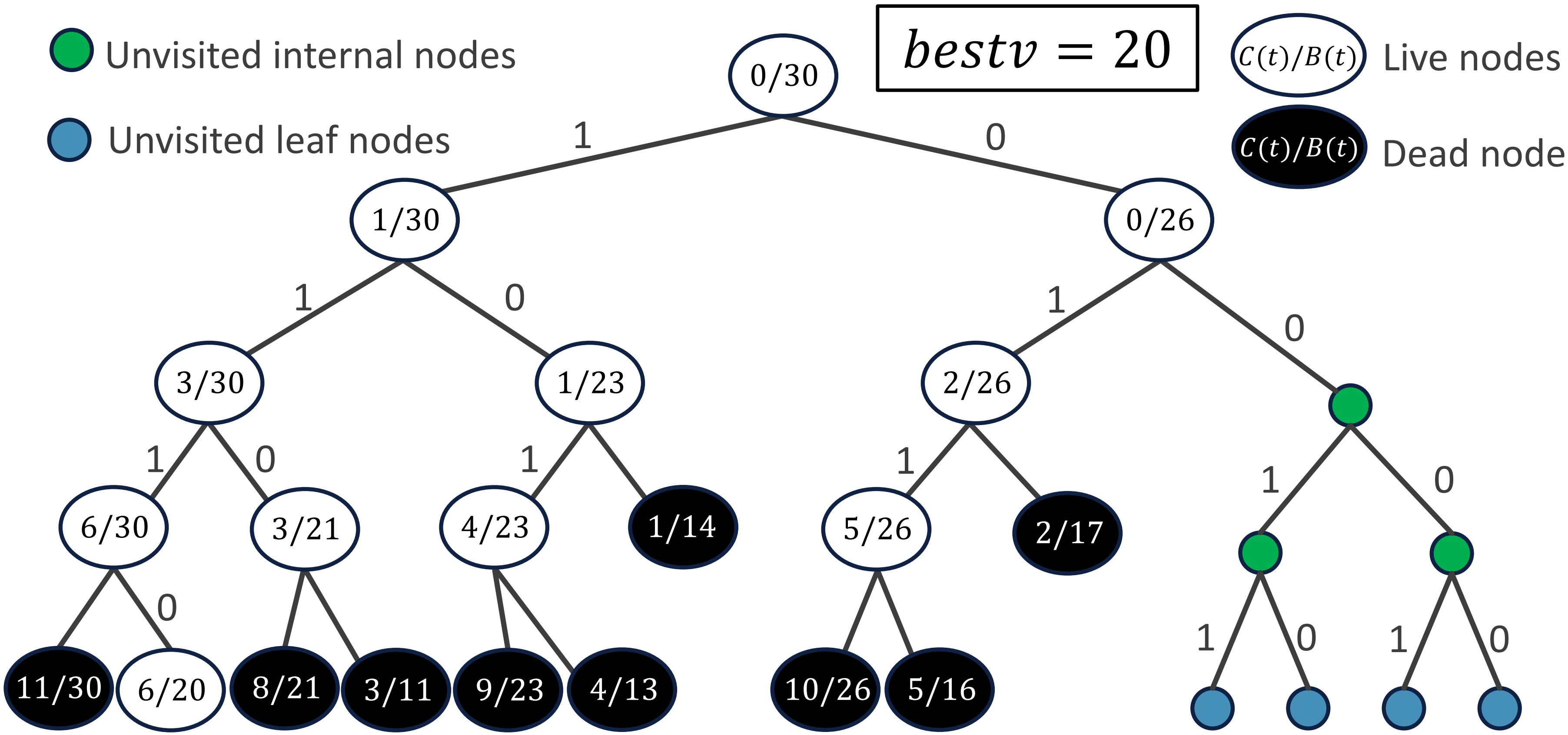
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



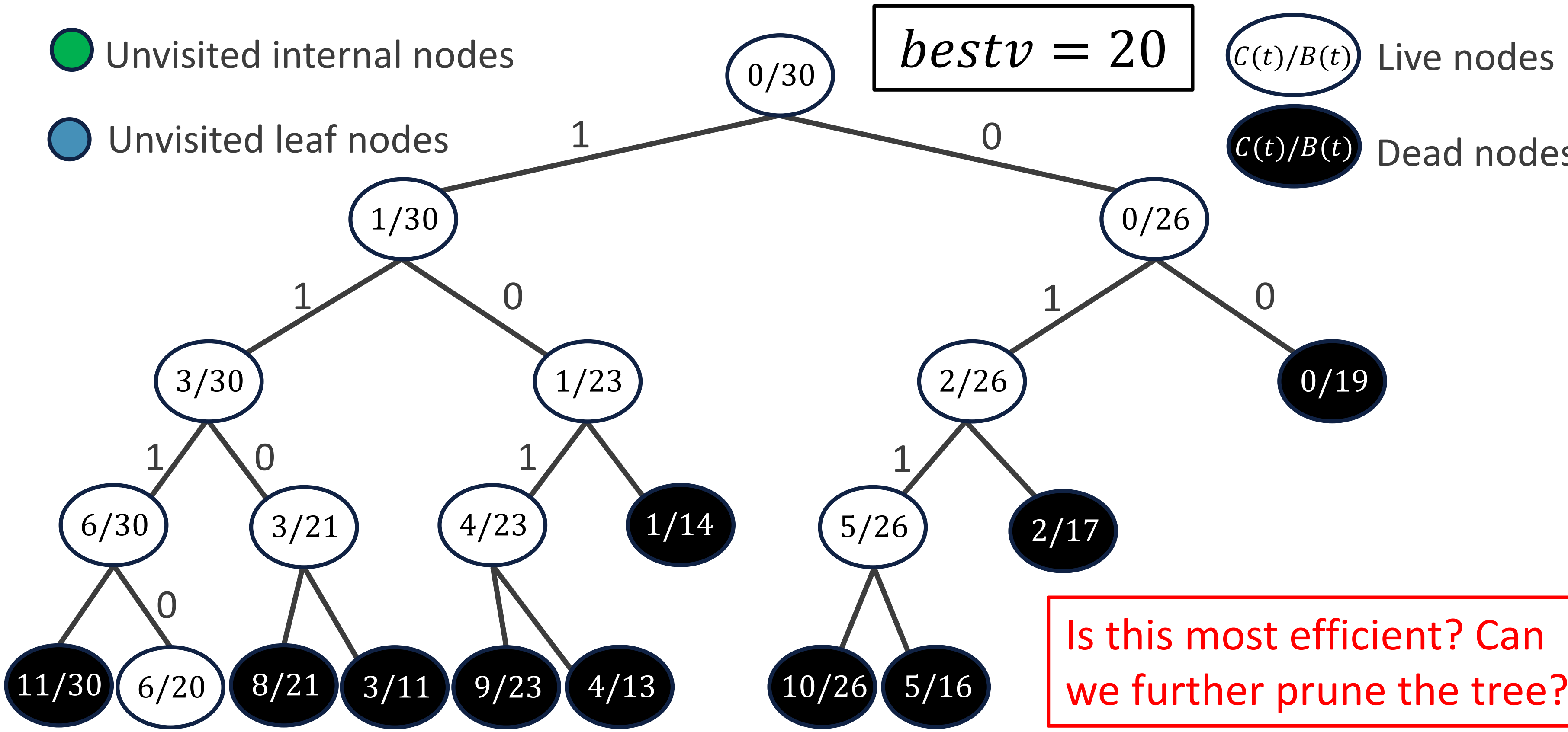
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

Example



Is this most efficient? Can we further prune the tree?

Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

0/1 Knapsack Problem

- Let's look back at the bounding function

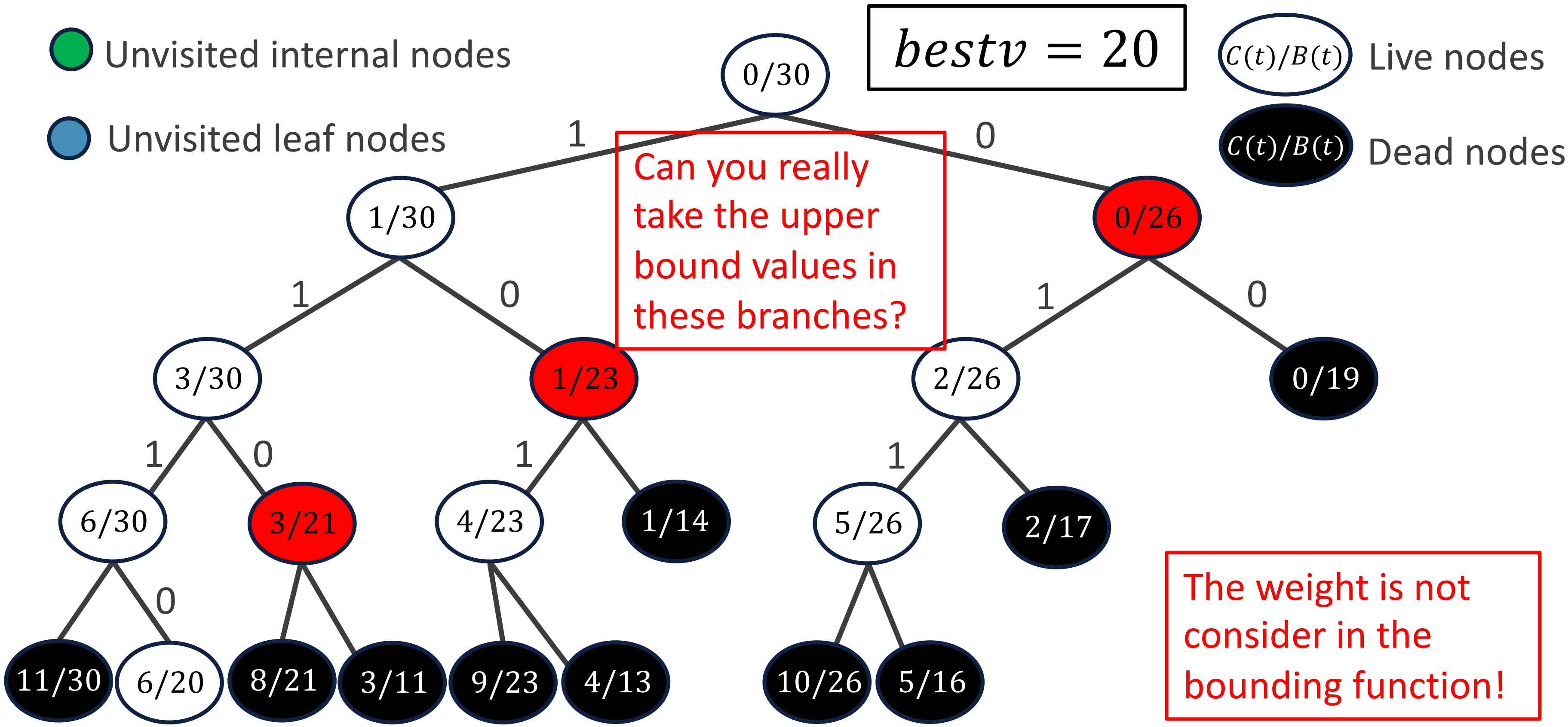
$$B(i) = cv(i) + r(i) \qquad cv(i) = \sum_{j=1}^{i} v_j x_j \qquad r(i) = \sum_{j=i+1}^{n} v_j$$

with the condition $B(i) \leq bestv$.

- What can we do if we want to prune more branches?

Make the bound tighter by decreasing the value of $B(i)$ (actually $r(i)$), because $C(i)$ is fixed at level i).

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$

0/1 Knapsack Problem

- Now, we consider the weight limit in the bounding function.
- Given the remaining capacity $W - cw(i)$, what is the maximum value can we get?
- We can use the following greedy strategy:
  - Take the most valuable remaining items until we can't take any more.
  - Take a fraction of the next item until fully loaded.
- It does not mean we can really take fraction of item. It is just the upper bound of the remaining value.

0/1 Knapsack Problem

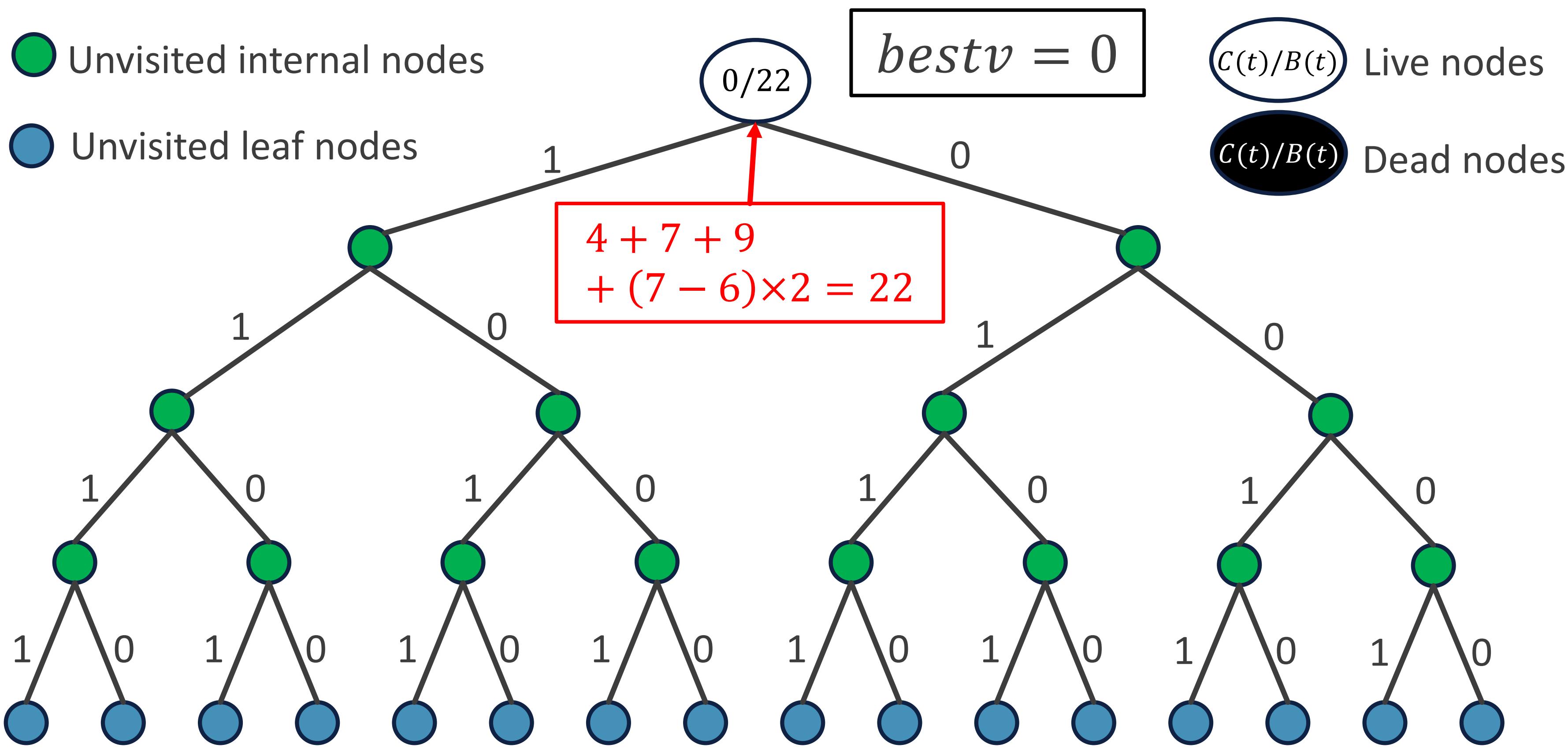
- First, sort the objects in decreasing order of value/weight ahead of time, namely

$$v_1/w_1 \geq v_2/w_2 \geq \cdots \geq v_n/w_n$$

- Now, we are at level i, which means we have made decision for the first i items.
- We continue to put from item $i+1$ until item k. When put item k in, the load exceeds W.
- Then we take a fraction of item k for the remaining capacity.

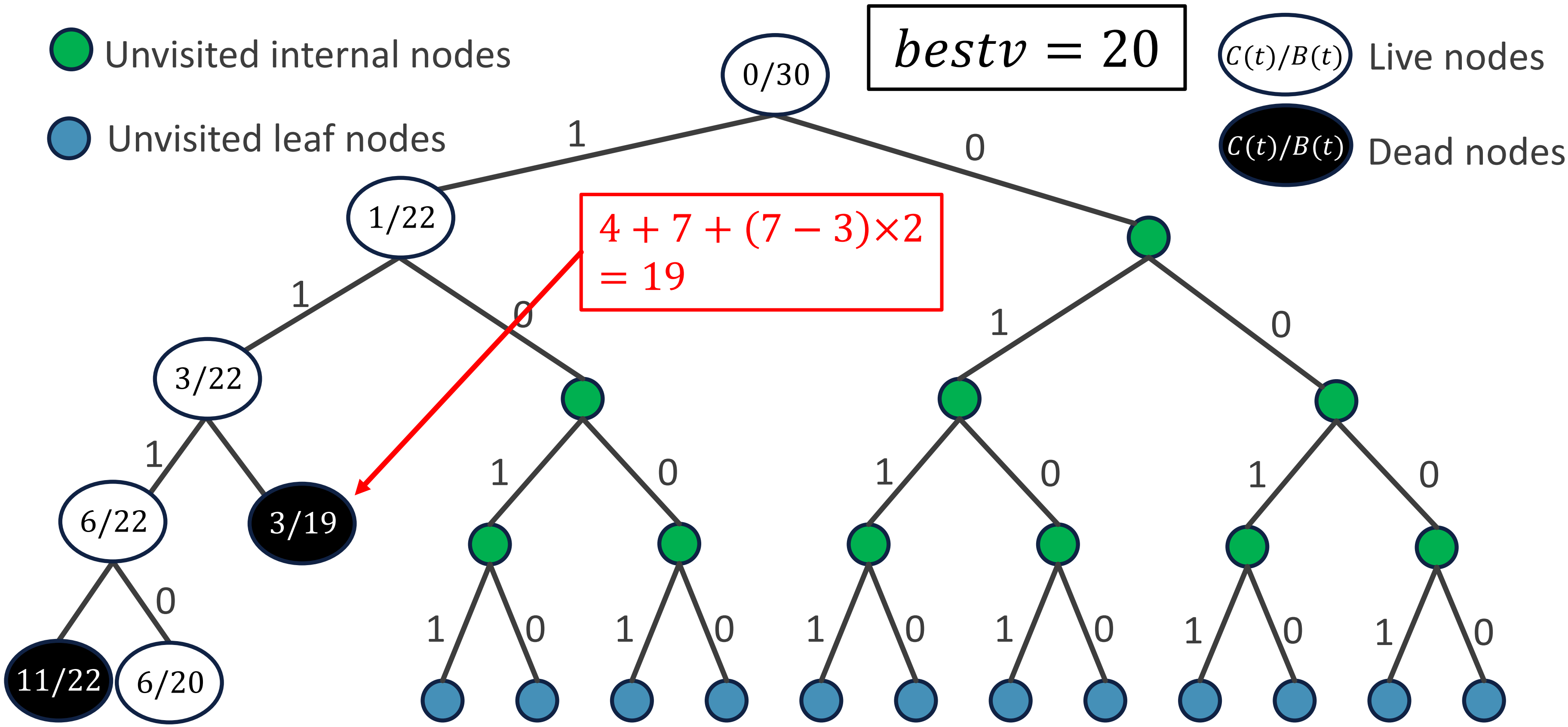
$$r(i) = \underbrace{\sum_{j=i+1}^{k-1} v_j}_{\text{Total value from item } i+1 \text{ to } k-1} + \underbrace{(W - cw(i) - \sum_{j=i+1}^{k-1} w_j\ )}_{\text{Capacity available for item } k}\underbrace{(\frac{v_k}{w_k})}_{\text{Value per unit weight for item } k}$$

Example



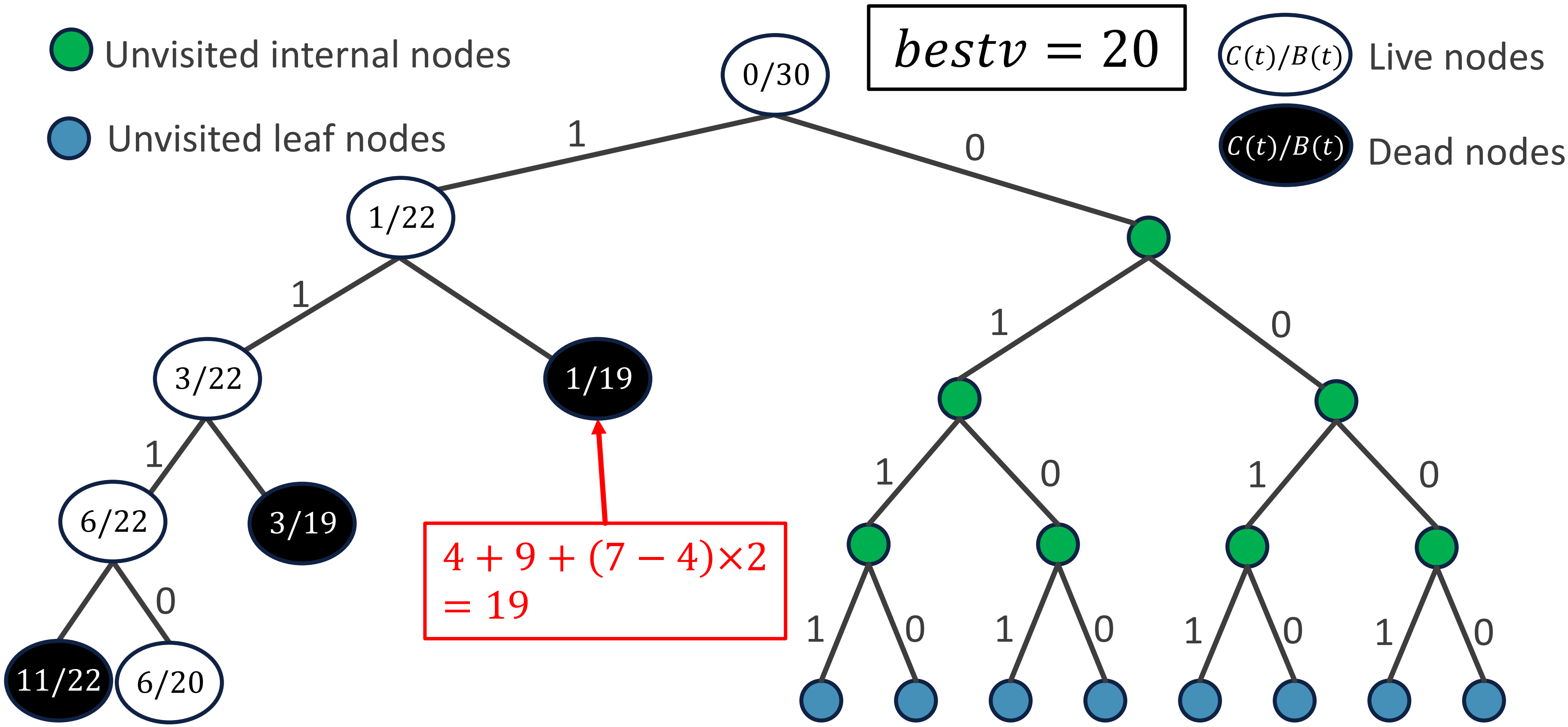
Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$, $v/w = [4, 3.5, 3, 2]$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$, $v/w = [4, 3.5, 3, 2]$

Example



Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$, $v/w = [4, 3.5, 3, 2]$

Example

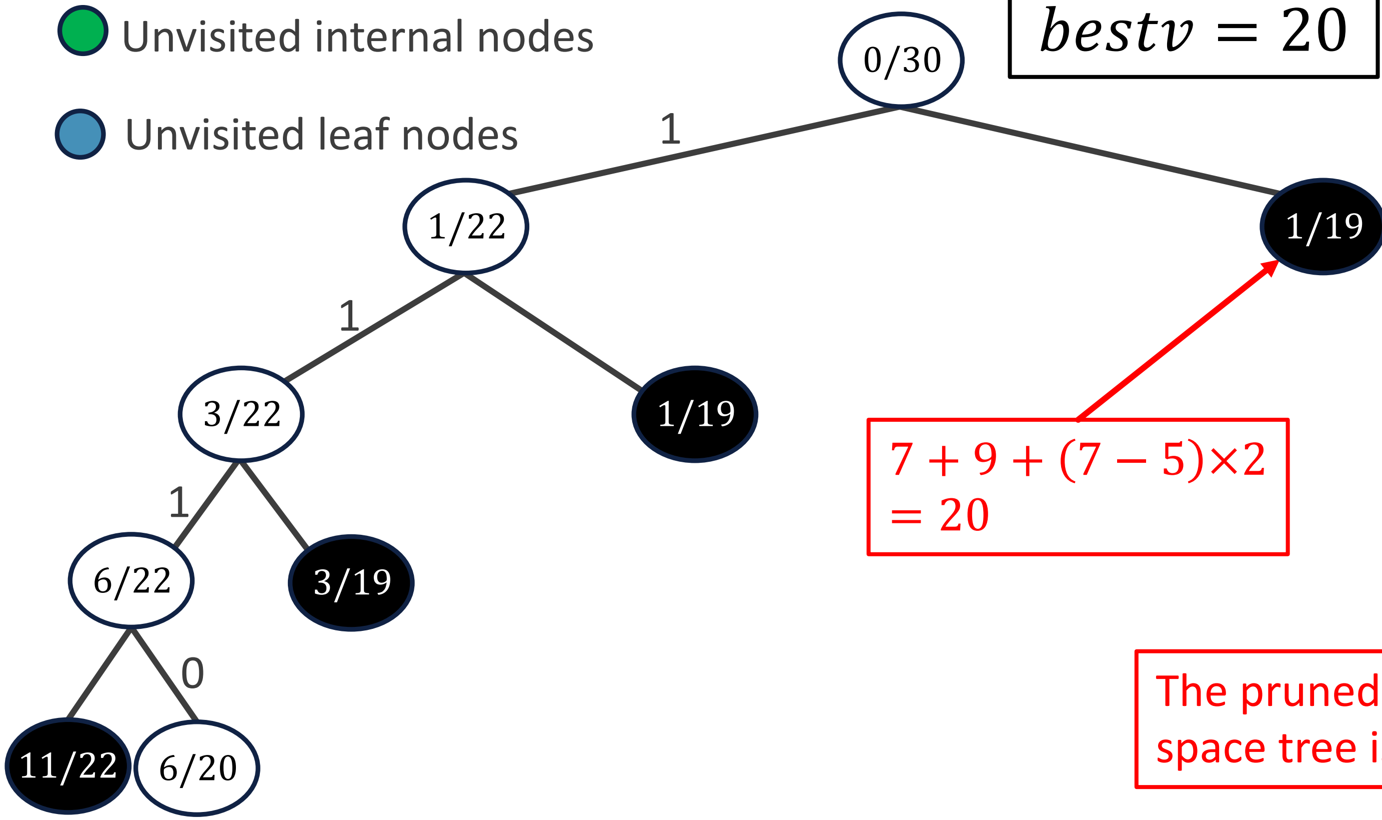
Unvisited internal nodes

Unvisited leaf nodes

$bestv = 20$

$C(t)/B(t)$ Live nodes

$C(t)/B(t)$ Dead nodes



The pruned solution space tree is much better!

Backtracking for $n = 4$, $v = [4,7,9,10]$, $w = [1,2,3,5]$, $W = 7$, $v/w = [4, 3.5, 3, 2]$

Pseudocode

BacktrackKnapsack(i)

if $i > n$ **then**
    if $cv > bestv$ **then**
        $bestv \leftarrow cv$
        for $j \leftarrow 1$ **to** n **do**
            $bestx[j] \leftarrow x[j]$
else
    if $C(i) \leq W$ **then** $x[i] \leftarrow 1$
        $cw \leftarrow cw + w[i]$; $cv \leftarrow cv + v[i]$;
        BacktrackKnapsack($i + 1$)
        $cw \leftarrow cw - w[i]$; $cv \leftarrow cv - v[i]$;
    if $B(i) > bestv$ **then** $x[i] \leftarrow 0$
        BacktrackKnapsack($i + 1$)

We don't record the remaining value here and leave it in $B(i)$.

Nothing special compared with container loading problem. Just separate cw and cv.

Pseudocode

$r(i)$

$rw \leftarrow W - cw$ Remaining capacity

$b \leftarrow cv$ Total value

while $i + 1 \leq n$ and $w[i+1] \leq rw$ **do** Loop until we can't take the whole item $i + 1$

  $rw \leftarrow rw - w[i+1]$

  $b \leftarrow b + v[i+1]$

  $i \leftarrow i + 1$ Take a fraction of item $i + 1$

if $i + 1 \leq n$ **then** $b \leftarrow b + v[i+1]/w[i+1] \times rw$

return b

Classroom Exercise

- Draw the pruned solution space tree of 0/1 knapsack problem for the following problem instance:

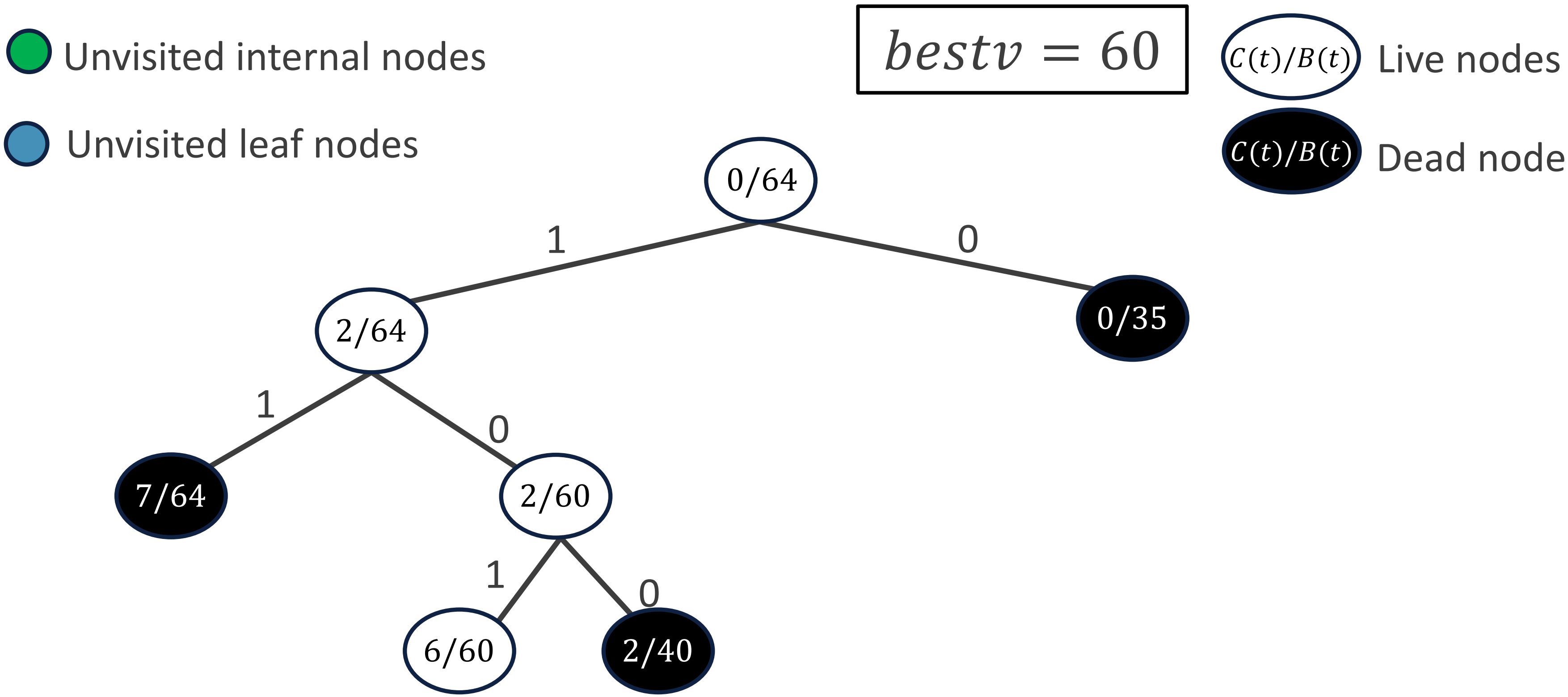
$$n = 3, v = [4,3,1], w = [2,5,5], W = 6$$

Classroom Exercise

- First, rank the item by their value per unit weight:

$n = 3, v = [40,30,20], w = [2,5,4], W = 6, v/w = [20,6,5]$

Classroom Exercise



Backtracking for $n = 3, v = [40,30,20], w = [2,5,4], W = 6, v/w = [20,6,5]$

n QUEEN PROBLEM

n Queen Problem

- The goal of n queen problem (n皇后问题) is to position n queens on an $n \times n$ chessboard so that no two queens threaten each other.
  - No two queens may be in the same row, column, or diagonal.

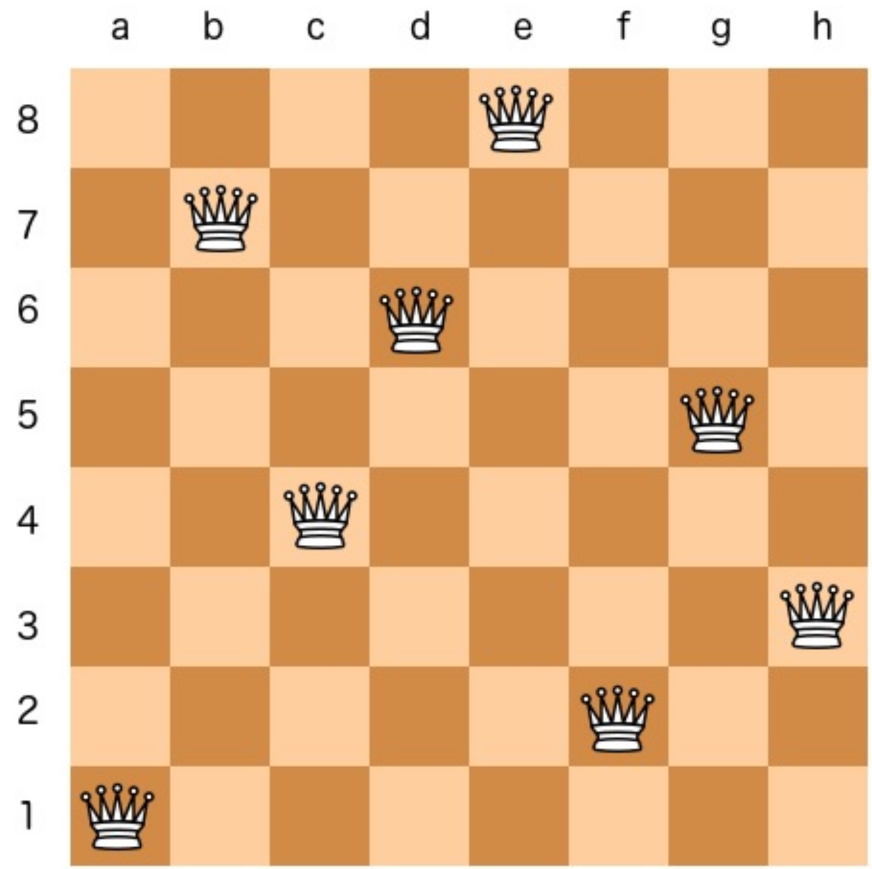


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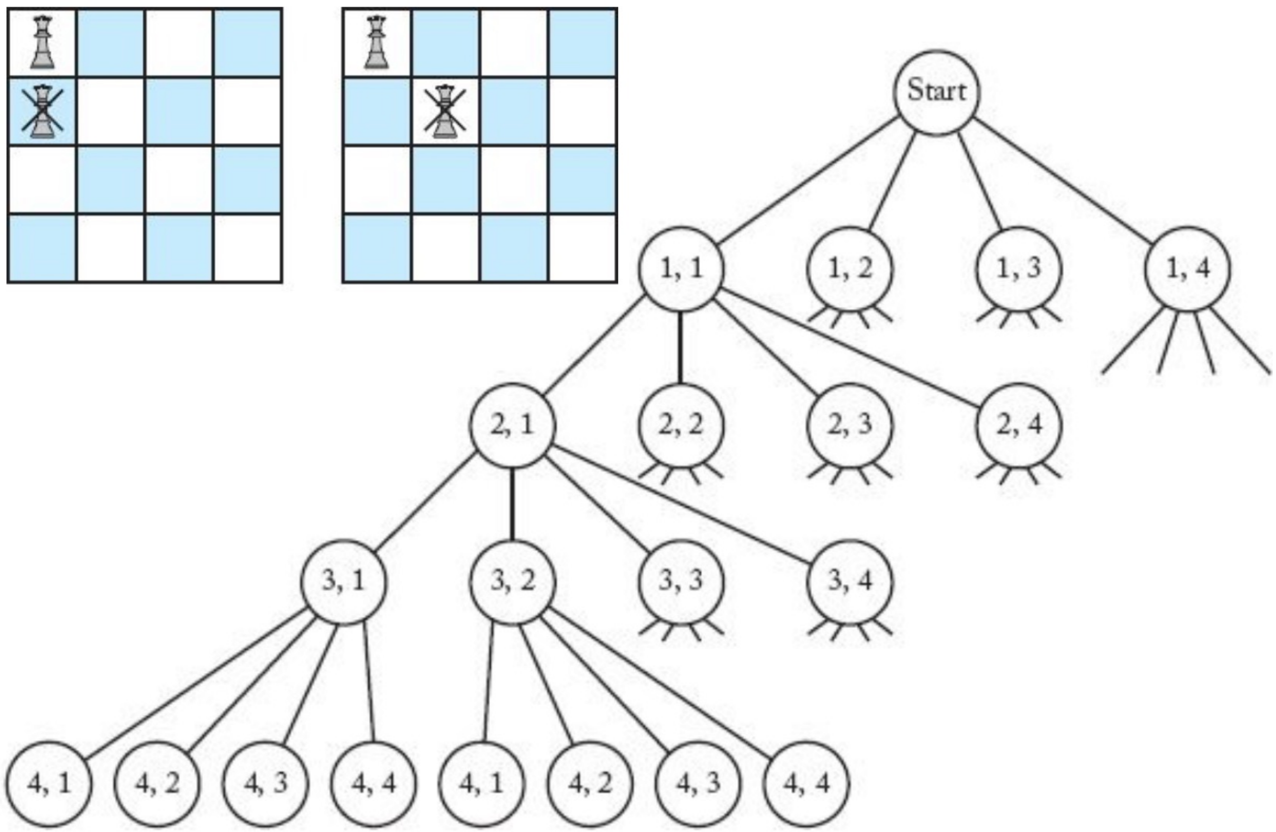
Image source: https://medium.com/swlh/how-many-solutions-does-the-n-queens-problem-have-e8da5d45a34c

n Queen Problem

- What is the size of solution space for the ith queen?
- $n^2 - i + 1$? It is too large. We can limit it by considering the constraint.
  - Because two queens can't be put in the same row, we directly put each queen in different row.
  - Now, the solution space for the ith queen is n.
  - Thus, the constraint function only needs to check if two queens are in the same column or diagonal.

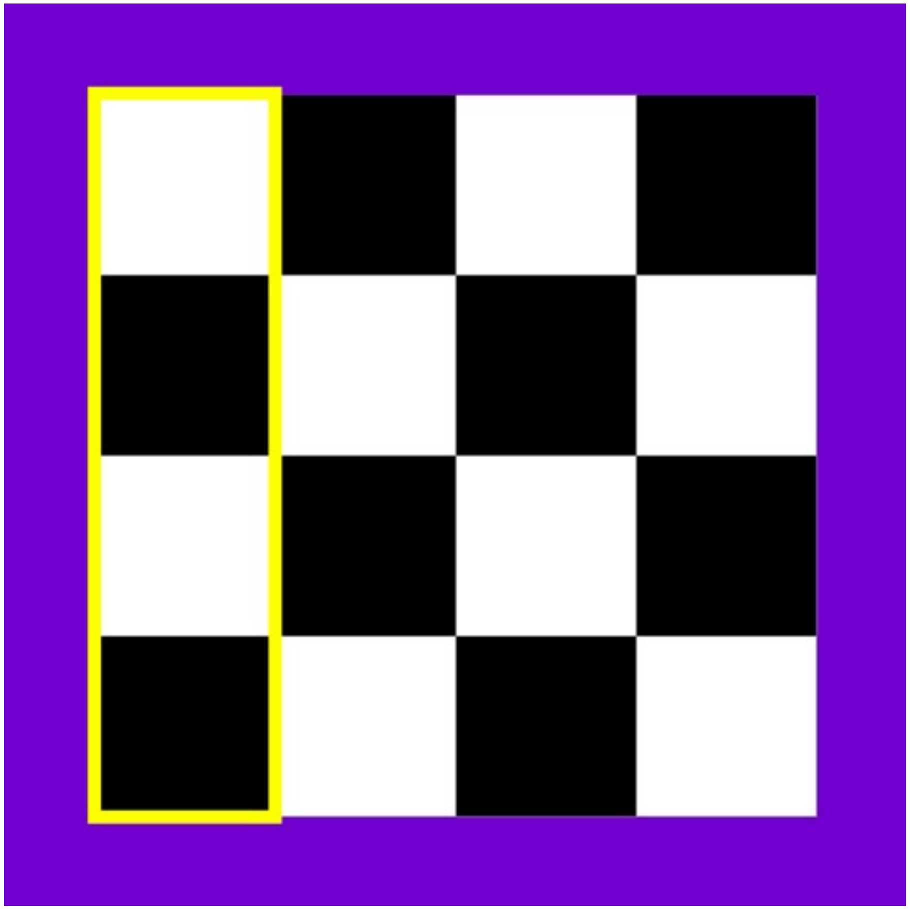
n Queen Problem



The solution space tree for $n = 4$

Image source: Figure 5.2-5.3, Richard E. Neapolitan, Foundations of Algorithms (5th Edition), Jones & Bartlett Learning, 2014

n Queen Problem



What is the constraint function?

Image source: https://meetwithbudhi.wordpress.com/2019/09/16/n-queens-puzzle/

n Queen Problem

- The constraint function checks if the new added queen is in the same column, or along the same diagonal.
- Now, we know that the ith queen is in the ith row. Let x_i be the column of the ith queen.
  - If the kth and jth queen are in the same column:

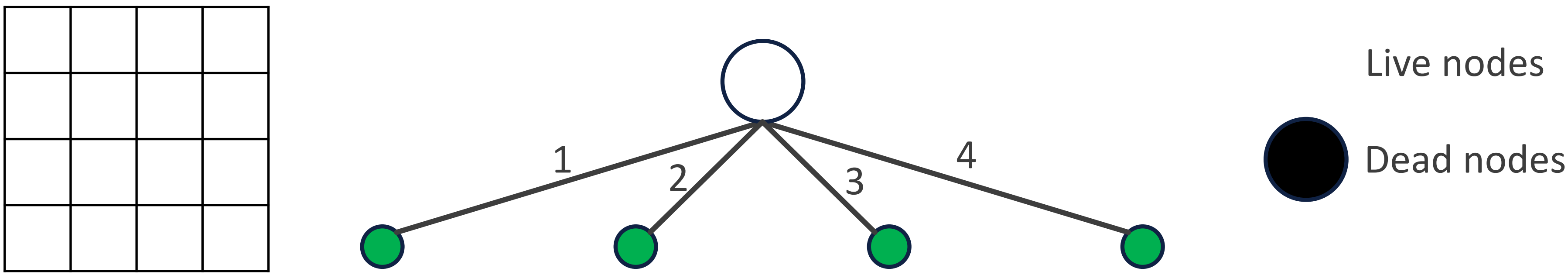
$$x_k = x_j$$

  - If the kth and jth queen are along the same diagonal:

$$x_k - x_j = k - j \quad \text{or} \quad x_k - x_j = j - k$$

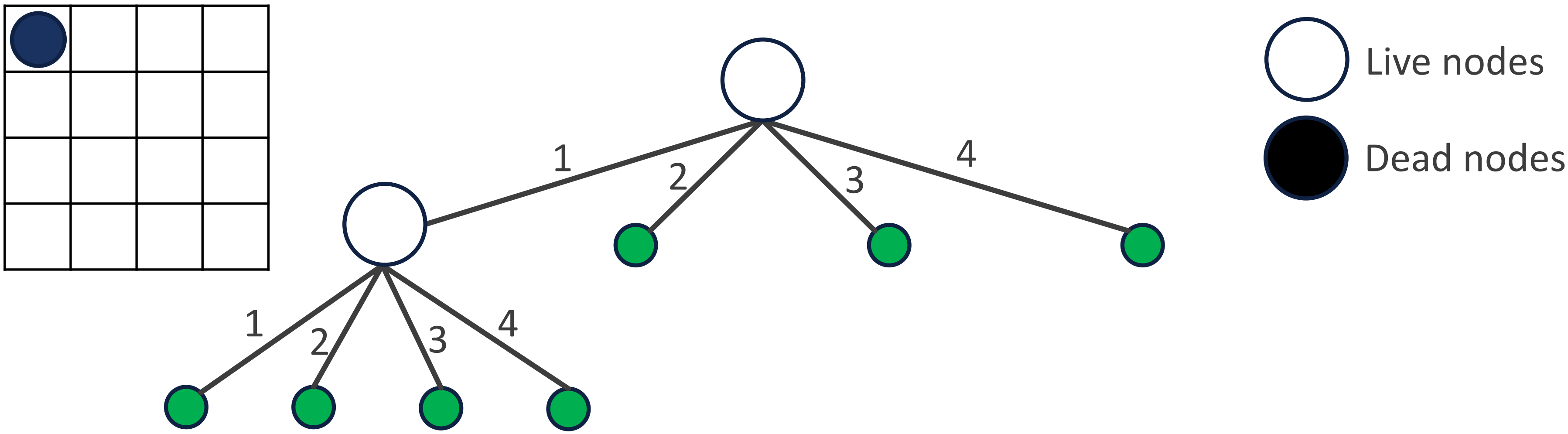
Namely: $|x_k - x_j| = |k - j|$.

Example



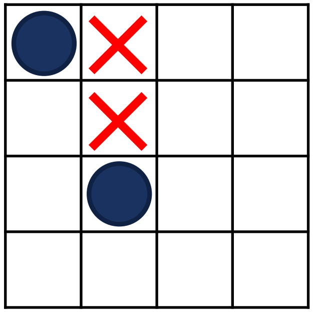
Backtracking for $n = 4$

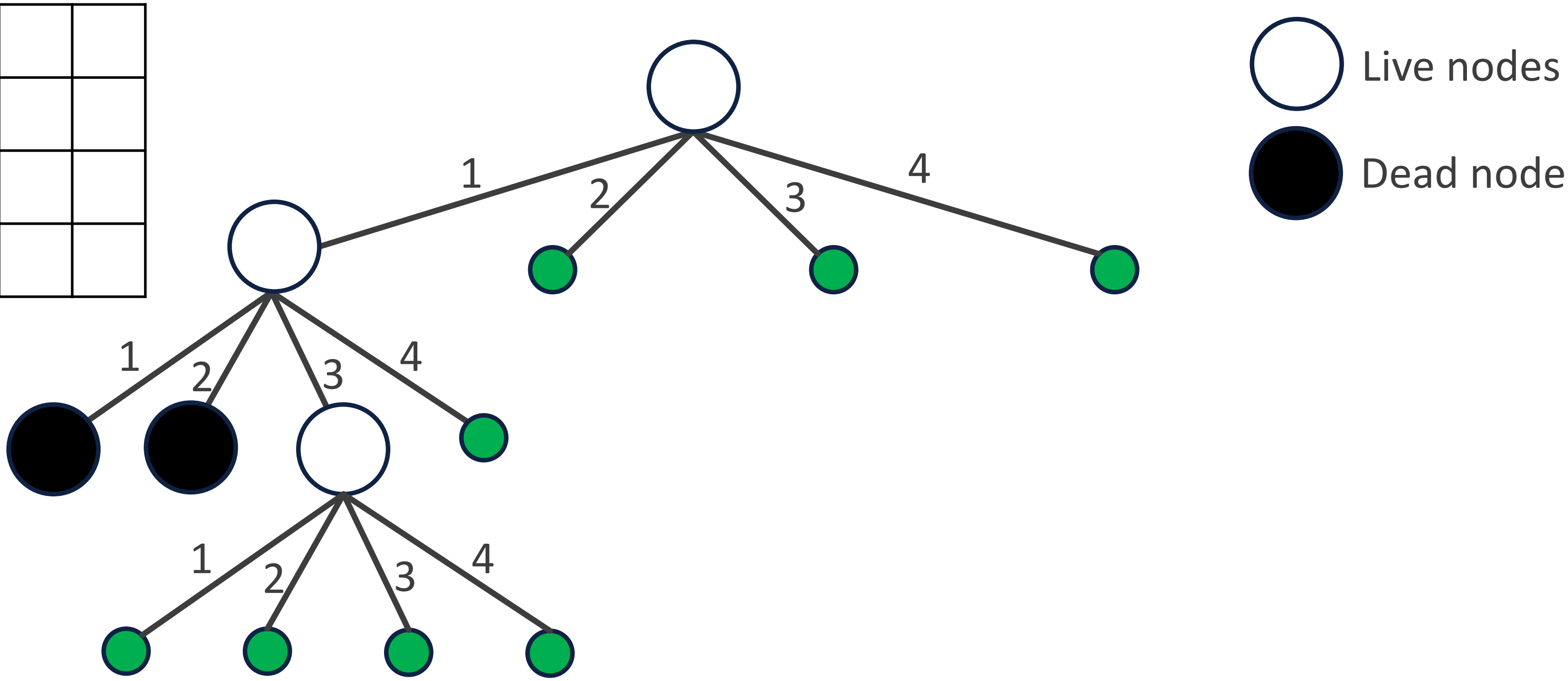
Example



Backtracking for $n = 4$

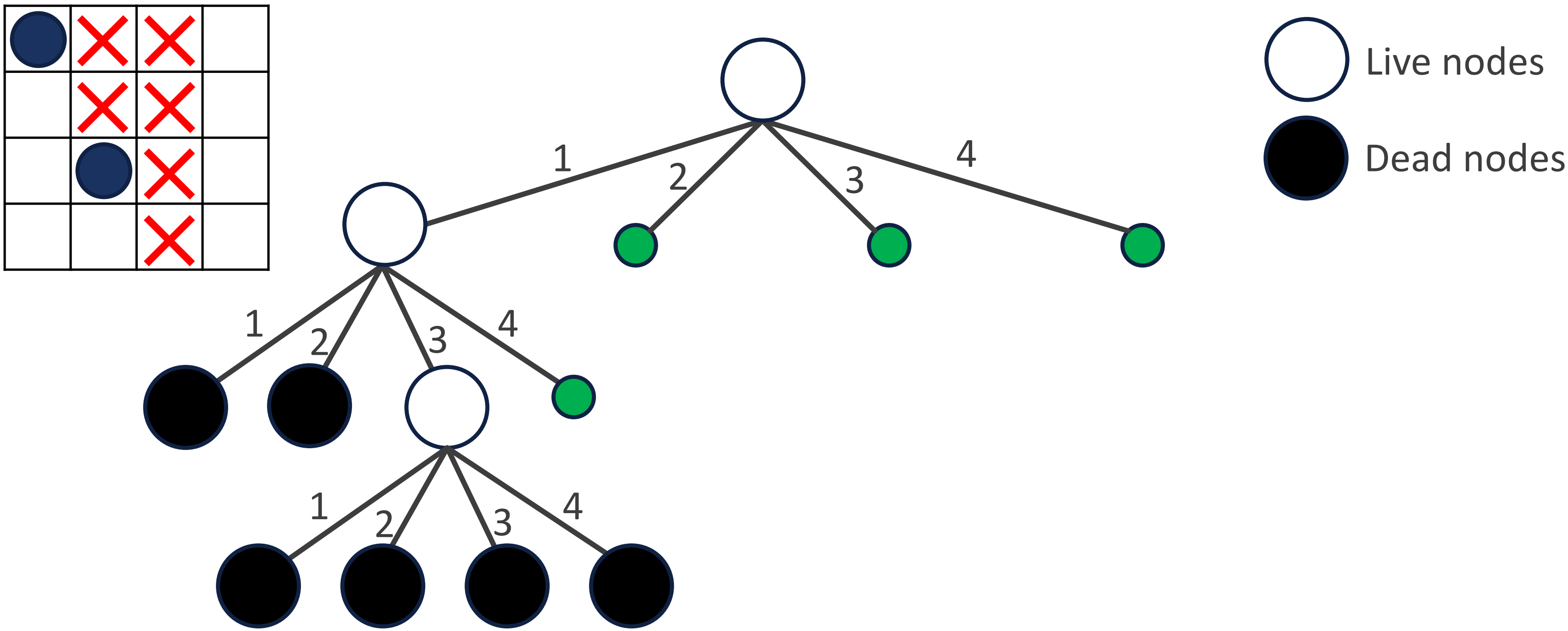
Example





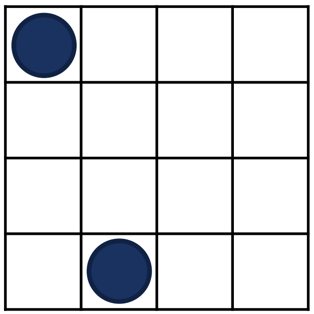
Backtracking for $n = 4$

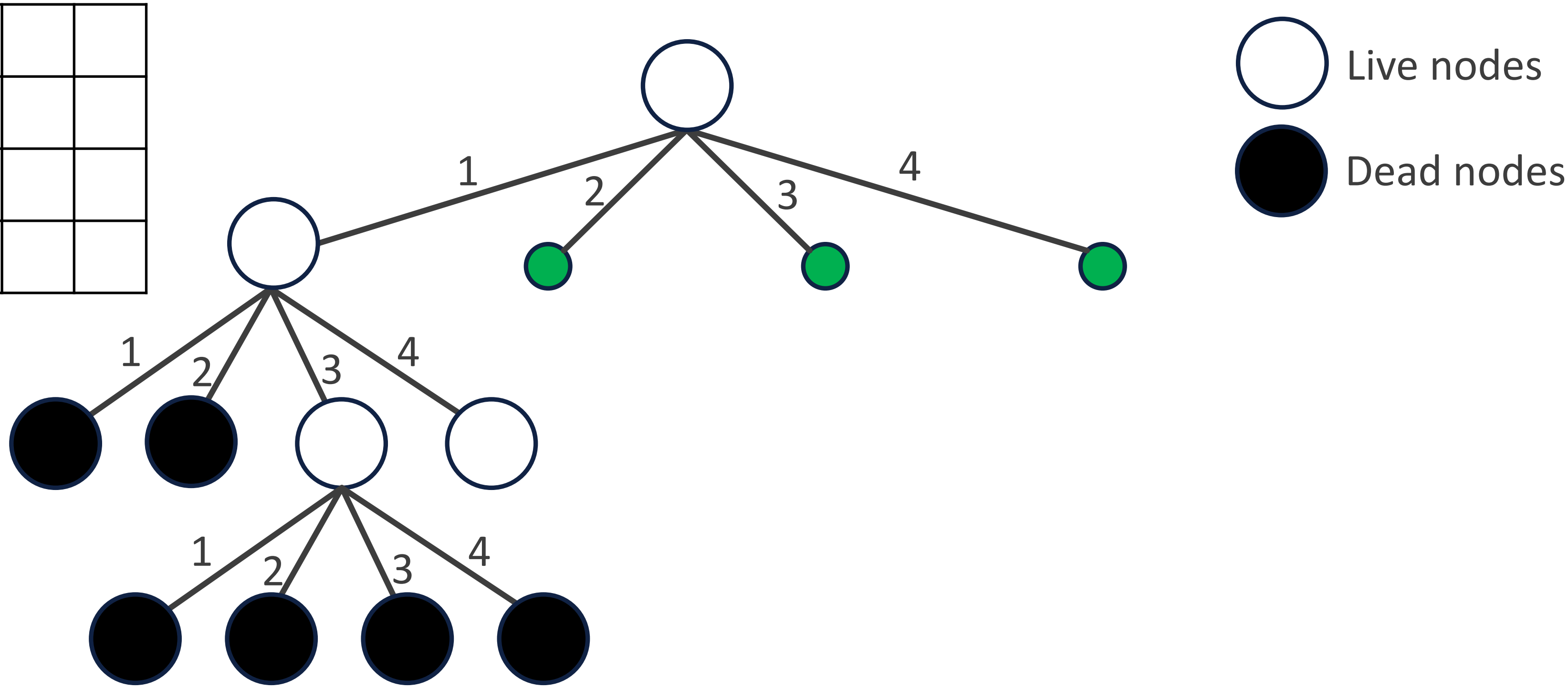
Example



Backtracking for $n = 4$

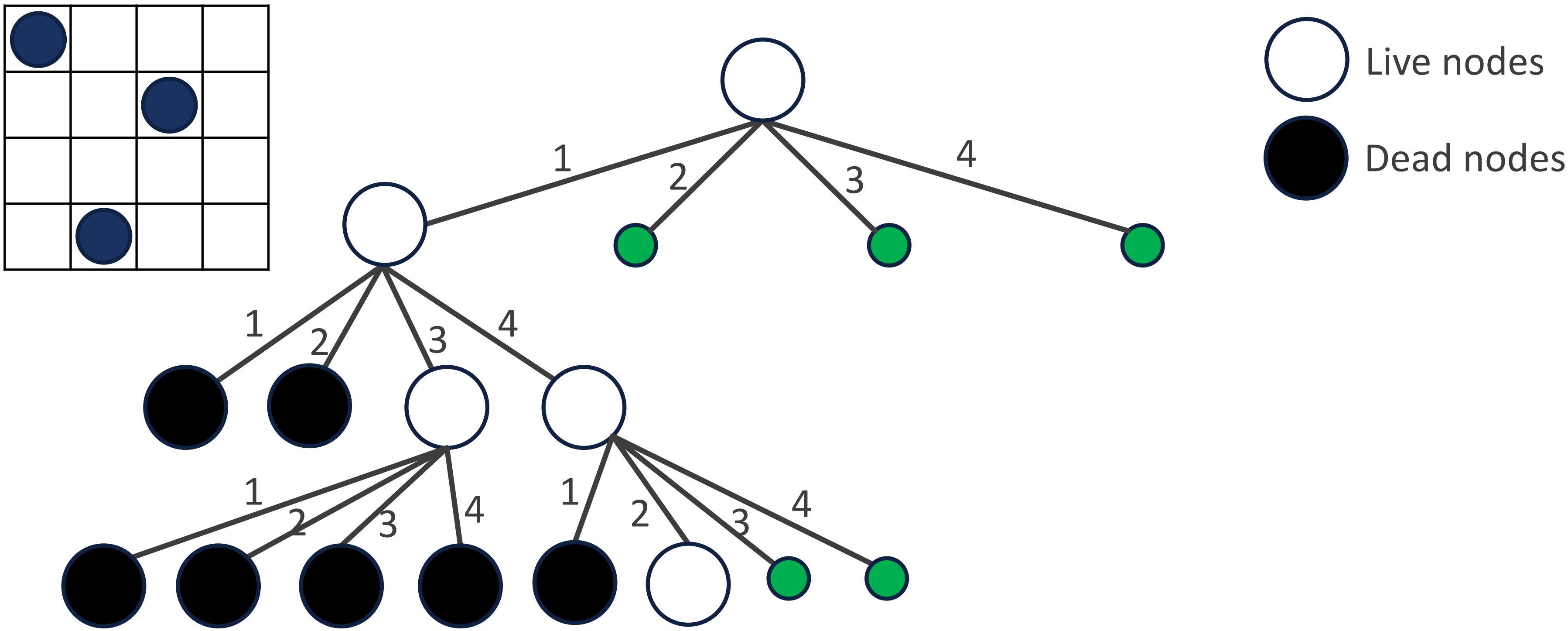
Example





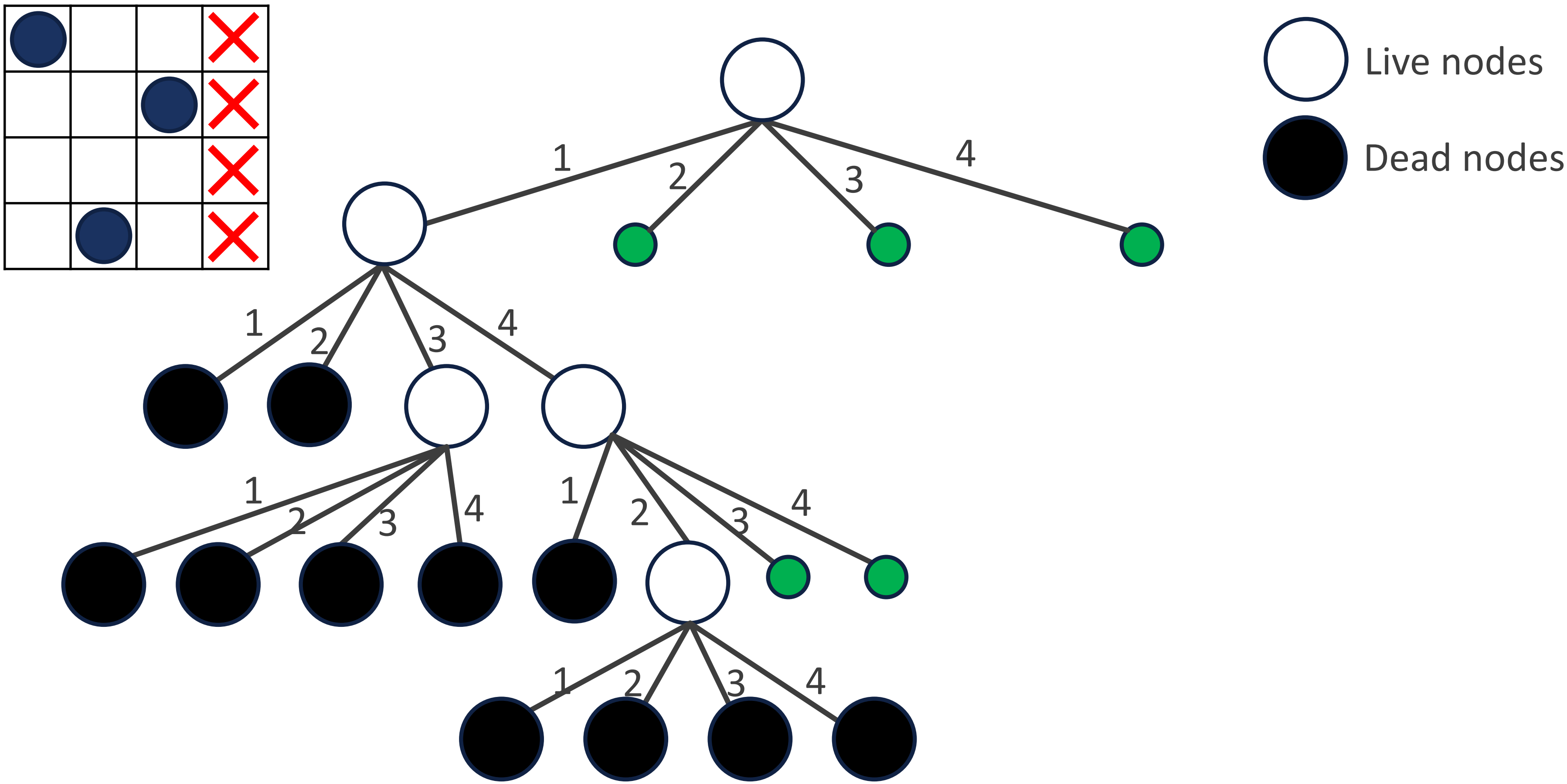
Backtracking for $n = 4$

Example



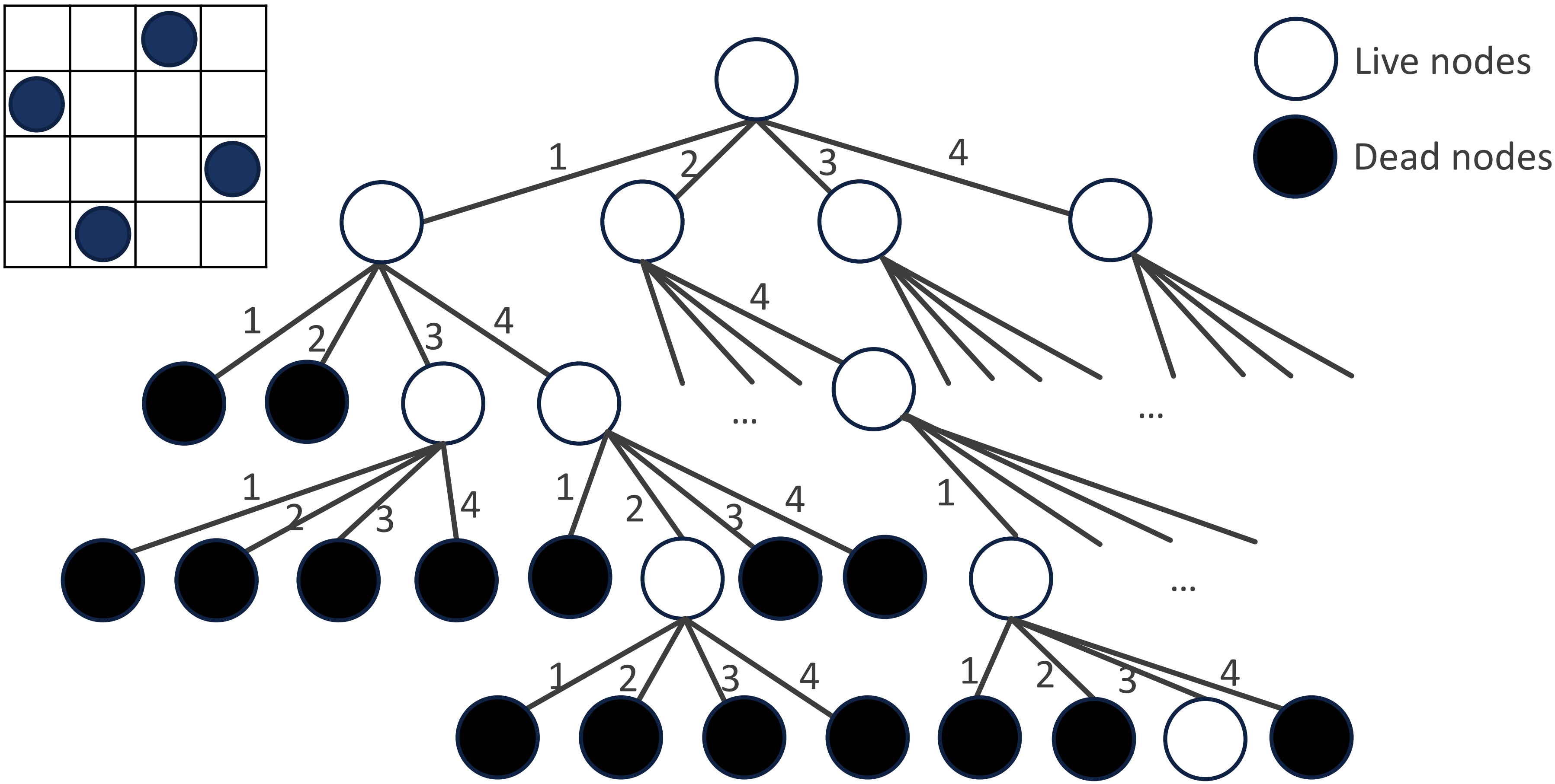
Backtracking for $n = 4$

Example



Backtracking for $n = 4$

Example



Backtracking for $n = 4$

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Computer Science Department of Xiamen University

Pseudocode

BacktrackNqueens()

$x[1] \leftarrow 0$
$k \leftarrow 1$
while $k > 0$ **do**
    while $x[k] \leq n-1$ **do**
        $x[k] \leftarrow x[k] + 1$
        if Place(k) = True **then**
            if $k = n$ **then** $SolNum \leftarrow SolNum + 1$
            else
                $k \leftarrow k + 1$
                $x[k] \leftarrow 0$
    $k \leftarrow k - 1$

Start from 0, increment in the loop, so the condition only checks $\leq n-1$

Number of valid solutions

Place(k)

for $j \leftarrow 1$ **to** $k-1$ **do**
    if $|k-j| = |x[k] - x[j]|$ or $x[j] = x[k]$ **then**
        return False
return True

No recursion is used here

Classroom Exercise

Write the pseudocode of the recursive version of n queen problem.

Classroom Exercise

```
RecursiveBacktrackNqueens(k)
    if Place(k) = True then
        if k = n then  SolNum ← SolNum + 1
        else
            for j ← 1 to n do
                x[k + 1] ← j
                RecursiveBacktrackNqueens(k + 1)
```

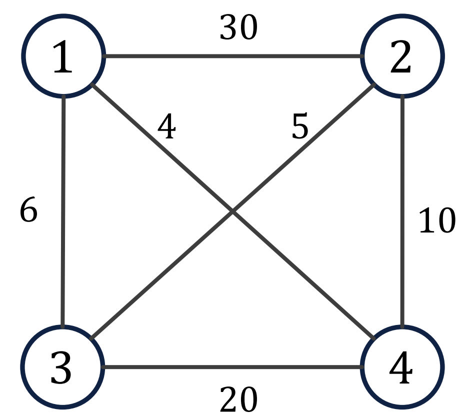
Start from 0

```
Place(k)
    for j ← 1 to k − 1 do
        if |k − j| = |x[k] − x[j]| or x[j] = x[k] then
            return False
    return True
```

TRAVELING SALESPERSON PROBLEM

Traveling Salesperson Problem

- Given an n vertex network (undirected or directed), traveling salesperson problem (旅行商问题, TSP) is to find a cycle of minimum cost that includes all n vertices.
  - Hamiltonian cycle with minimum cost.
- Any cycle that includes all n vertices of a network is called a tour. In TSP, we are to find a least-cost tour. For example:
  - Tour (1,2,4,3,1) costs 66.
  - Tour (1,4,3,2,1) costs 59.
  - Tour (1,3,2,4,1) costs 25, optimal.



Traveling Salesperson Problem

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Image source: https://www.programmersought.com/article/32324255027/

Traveling Salesperson Problem

- Since a tour is a cycle that includes all vertices, we may pick any vertex as the start (and hence the end).
  - Usually we use vertex 1 as the start and end vertex.
- Each tour is then described by the vertex sequence:

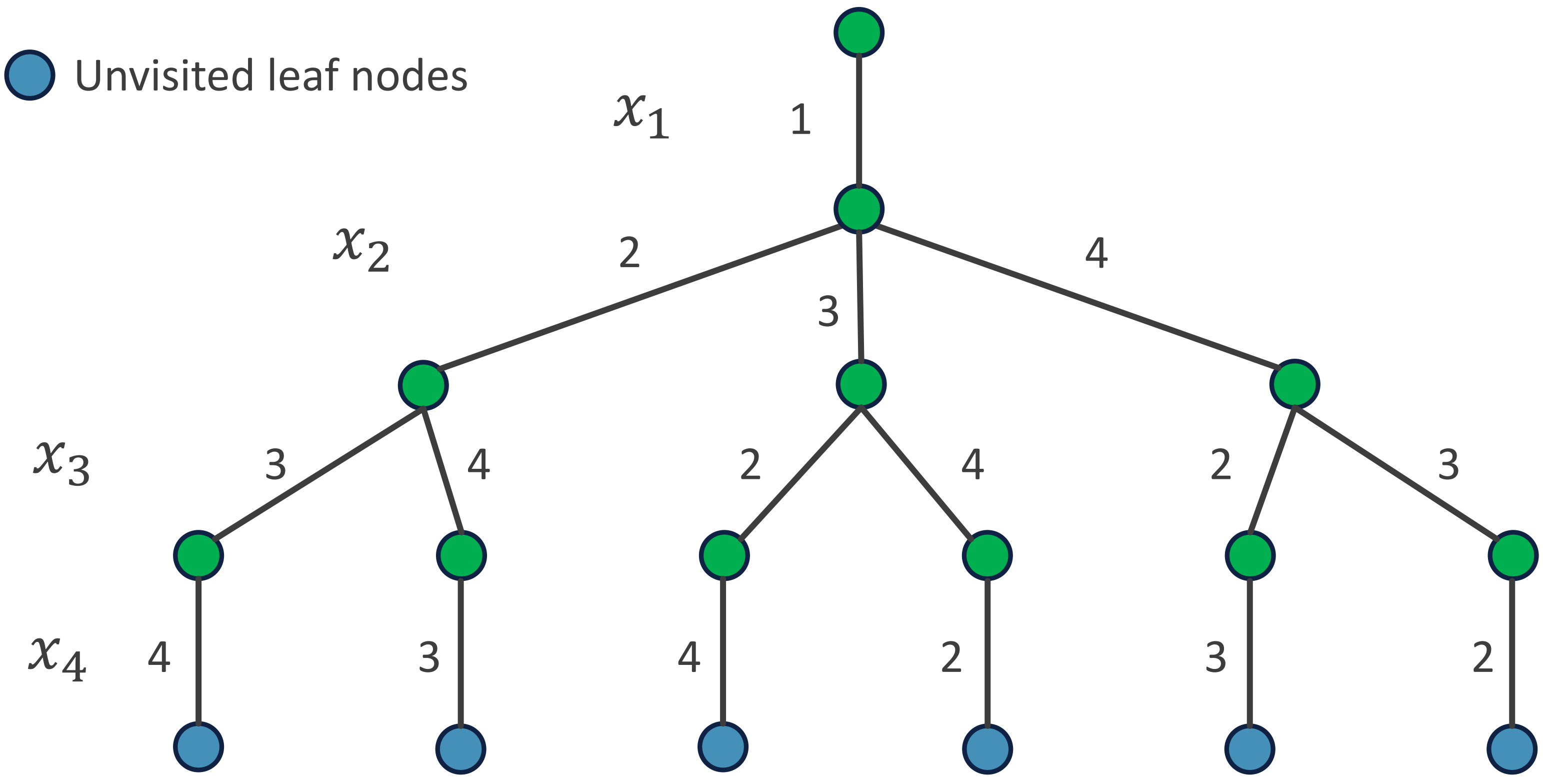
$$(1, x_2, \dots, x_n, 1)$$

  where $x_2, \dots, x_n$ is a permutation of $(2, 3, \dots, n)$.
- The possible tours may be described by a permutation tree in which each root-to-leaf path defines a tour.

Traveling Salesperson Problem

Unvisited internal nodes

Unvisited leaf nodes



Permutation tree for TSP when $n = 4$

Traveling Salesperson Problem

- $w[i, j]$ denotes the weight of vertex i and vertex j.
- $w[i, j] = \infty$ denotes no edge between vertex i and vertex j.
- $x[i]$ denotes the vertex to be searched.
- What are the constraint function and bounding function?

Traveling Salesperson Problem

- Constraint function $C(i)$ is to simply check if the next vertex is connected to the current vertex:

$$C(i) = w\big[x[i], x[j]\big]$$

Check if $C(i) \neq \infty$.

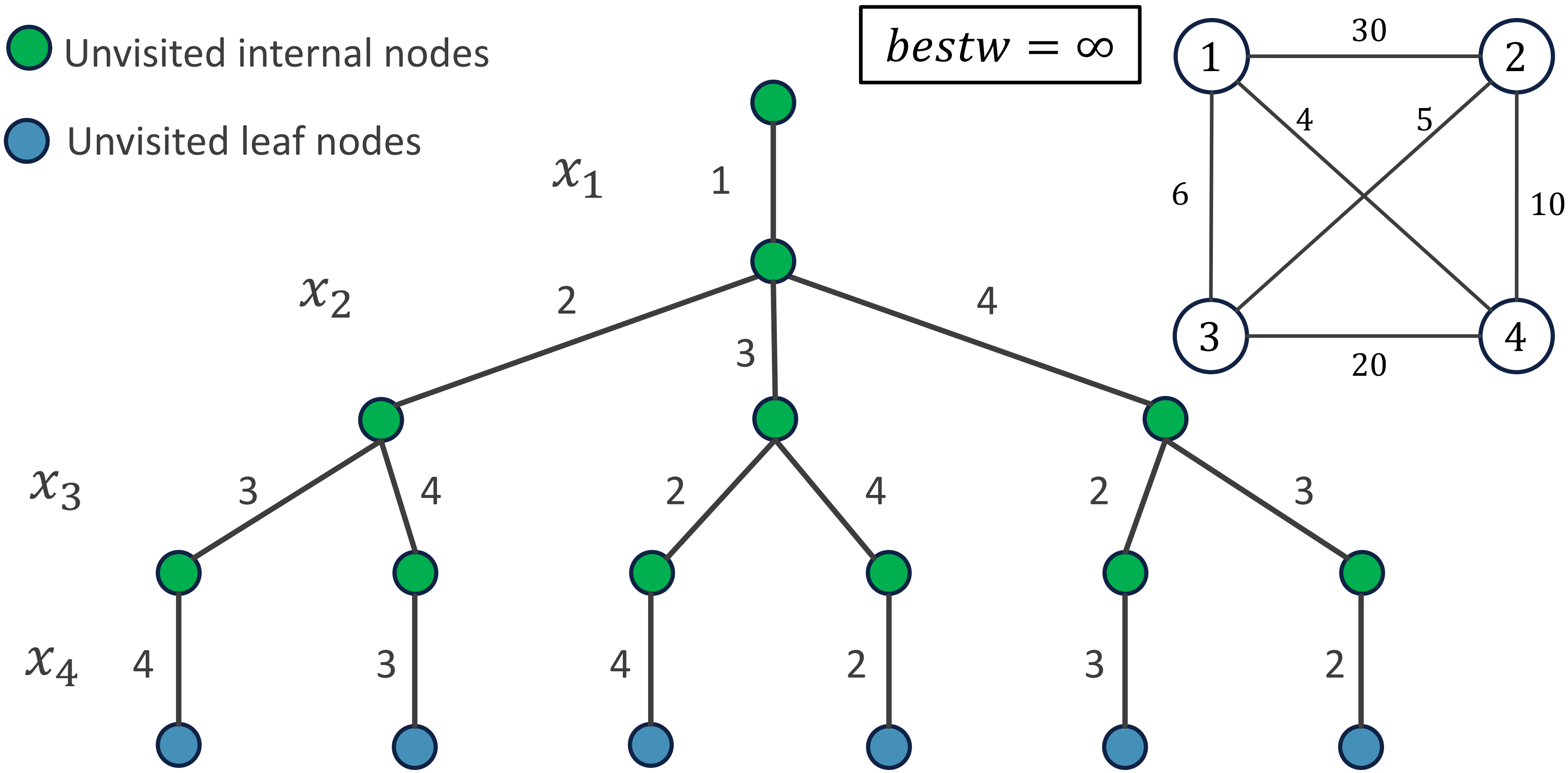
- Bounding function $B(i)$ is the total weight if we connect $x[i]$:

$$B(i) = cw(i-1) + w[x[i-1], x[i]]$$

$$cw(i) = \sum_{j=2}^{i} w[x[j-1], x[j]]$$

Check if $B(i) < bestw$.

Example



This graph is complete graph. Therefore the constraint function is useless.

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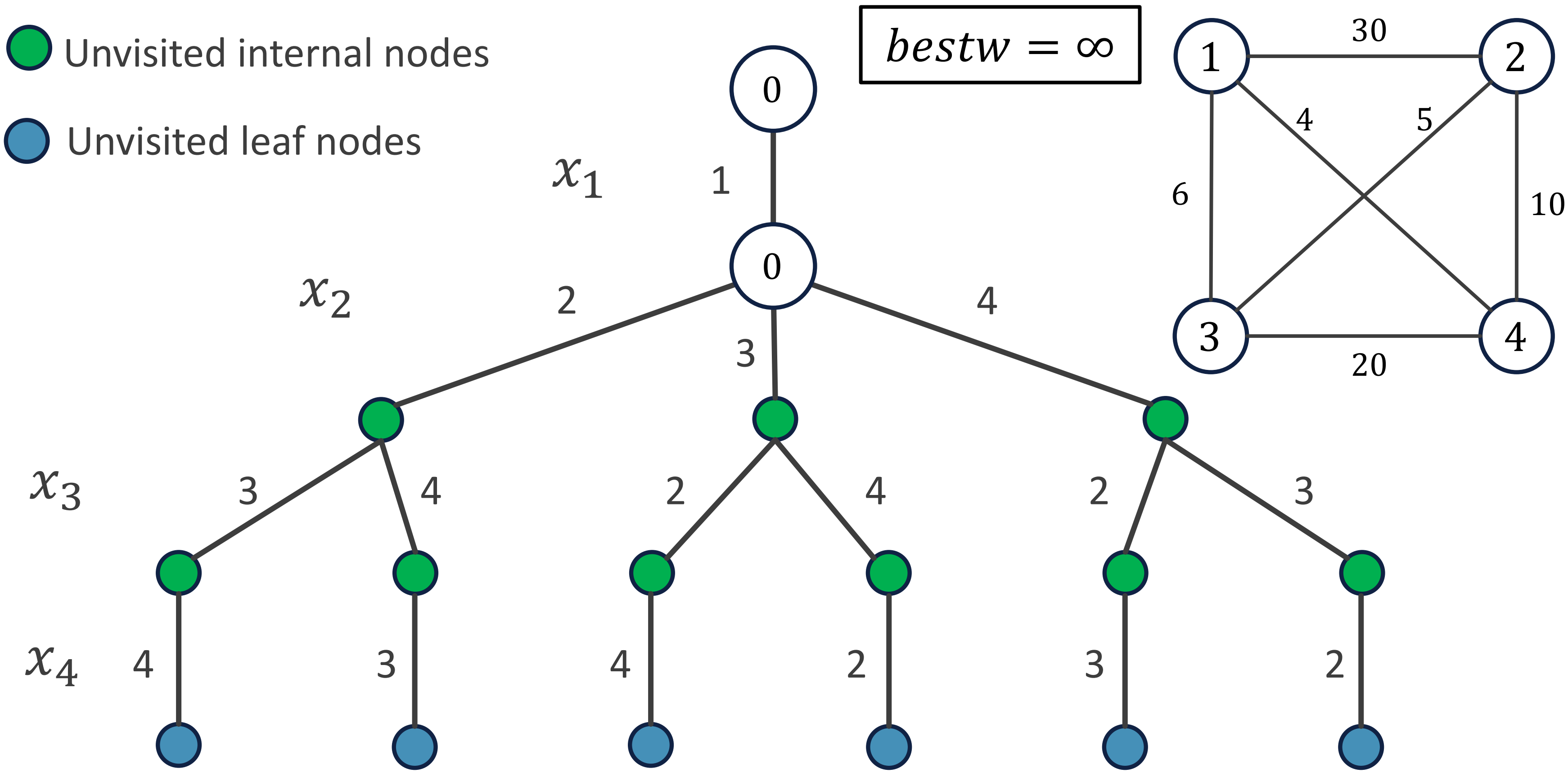
Example



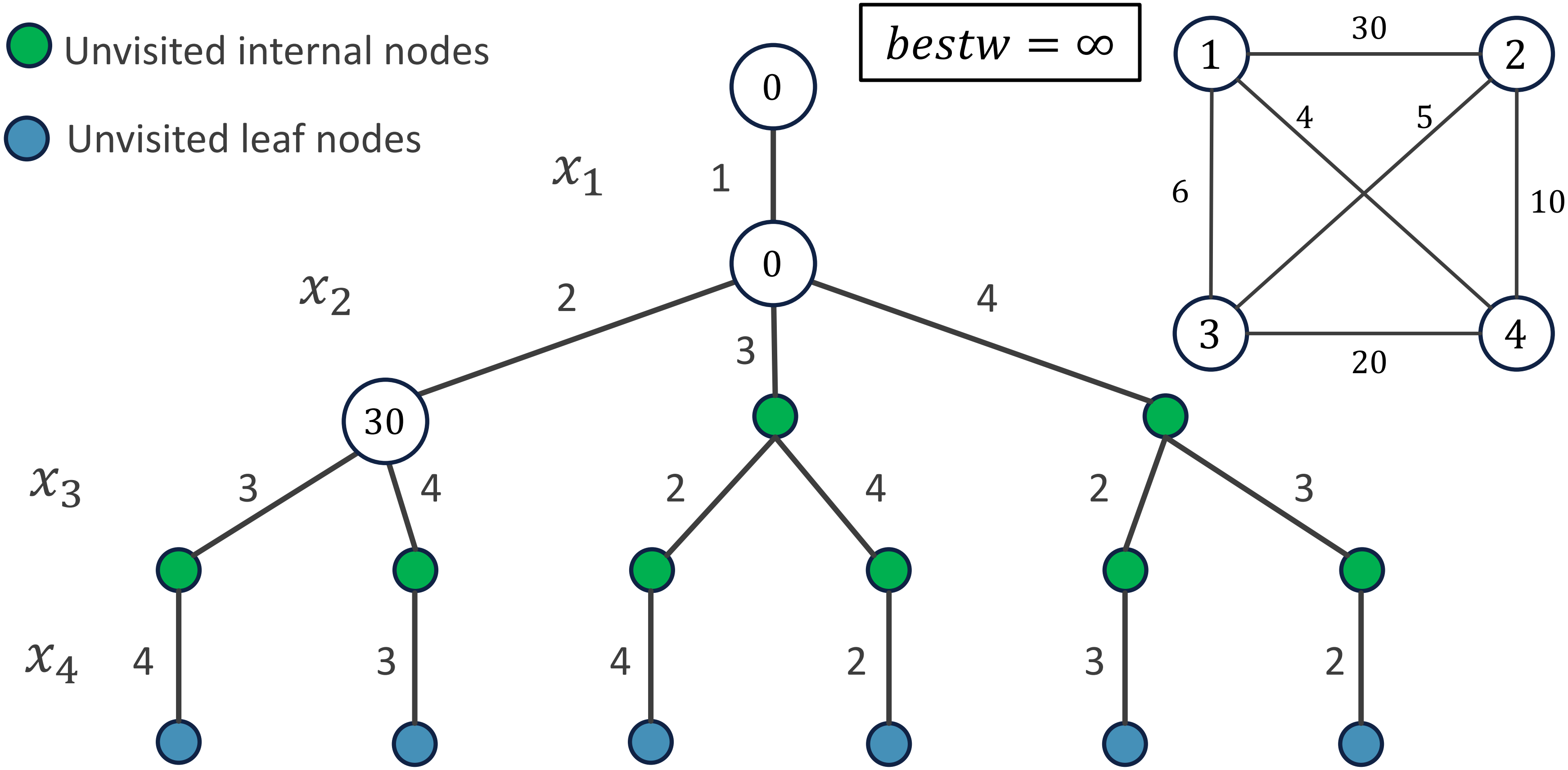
Unvisited internal nodes
Unvisited leaf nodes
$bestw = \infty$
0
1
0
x_1
x_2
x_3
x_4
2
3
4
3
4
2
4
2
3
4
3
4
2
3
2
1
2
3
4
30
4
5
6
10
20

Example

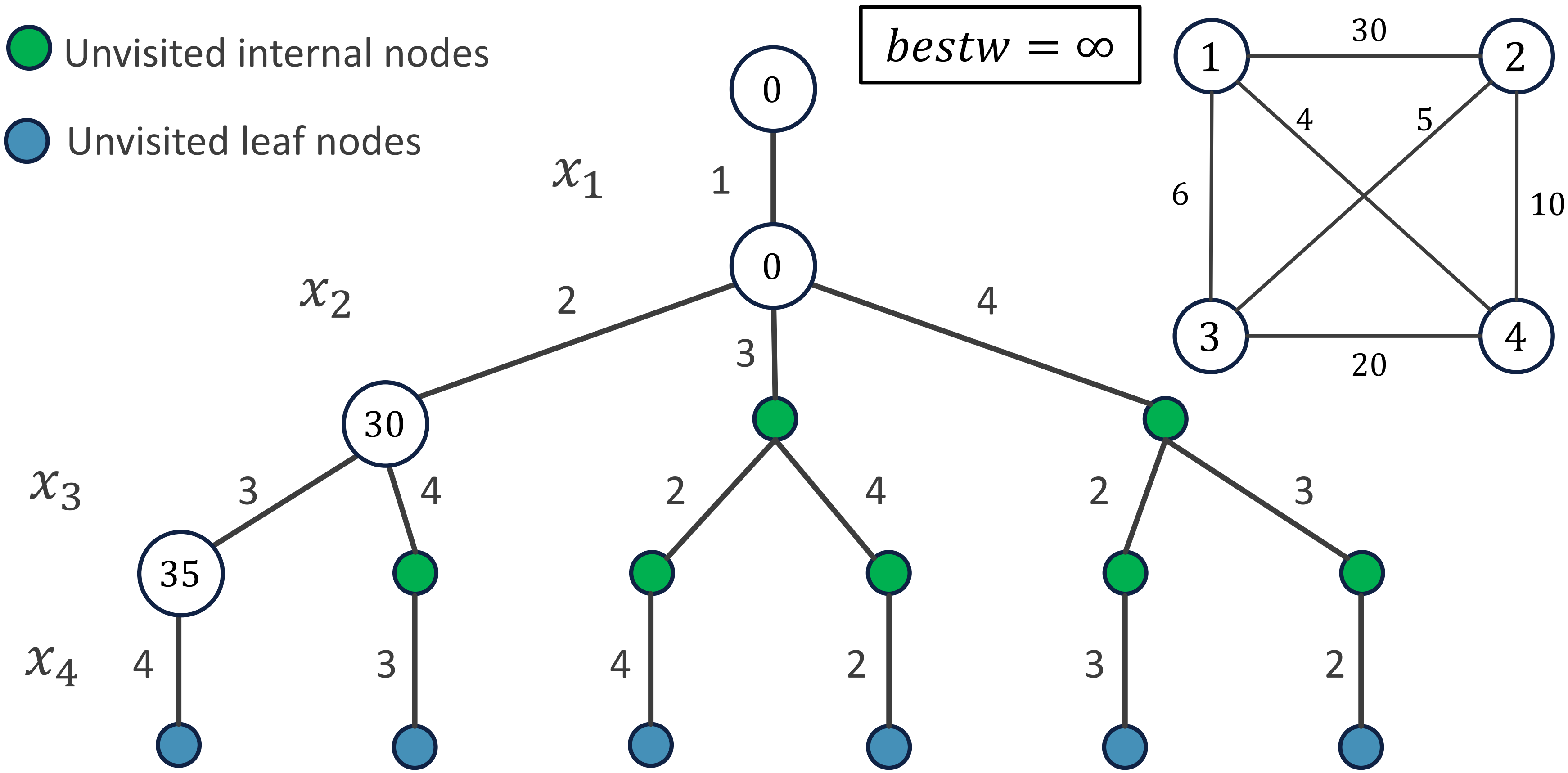
Unvisited internal nodes

Unvisited leaf nodes

$bestw = \infty$

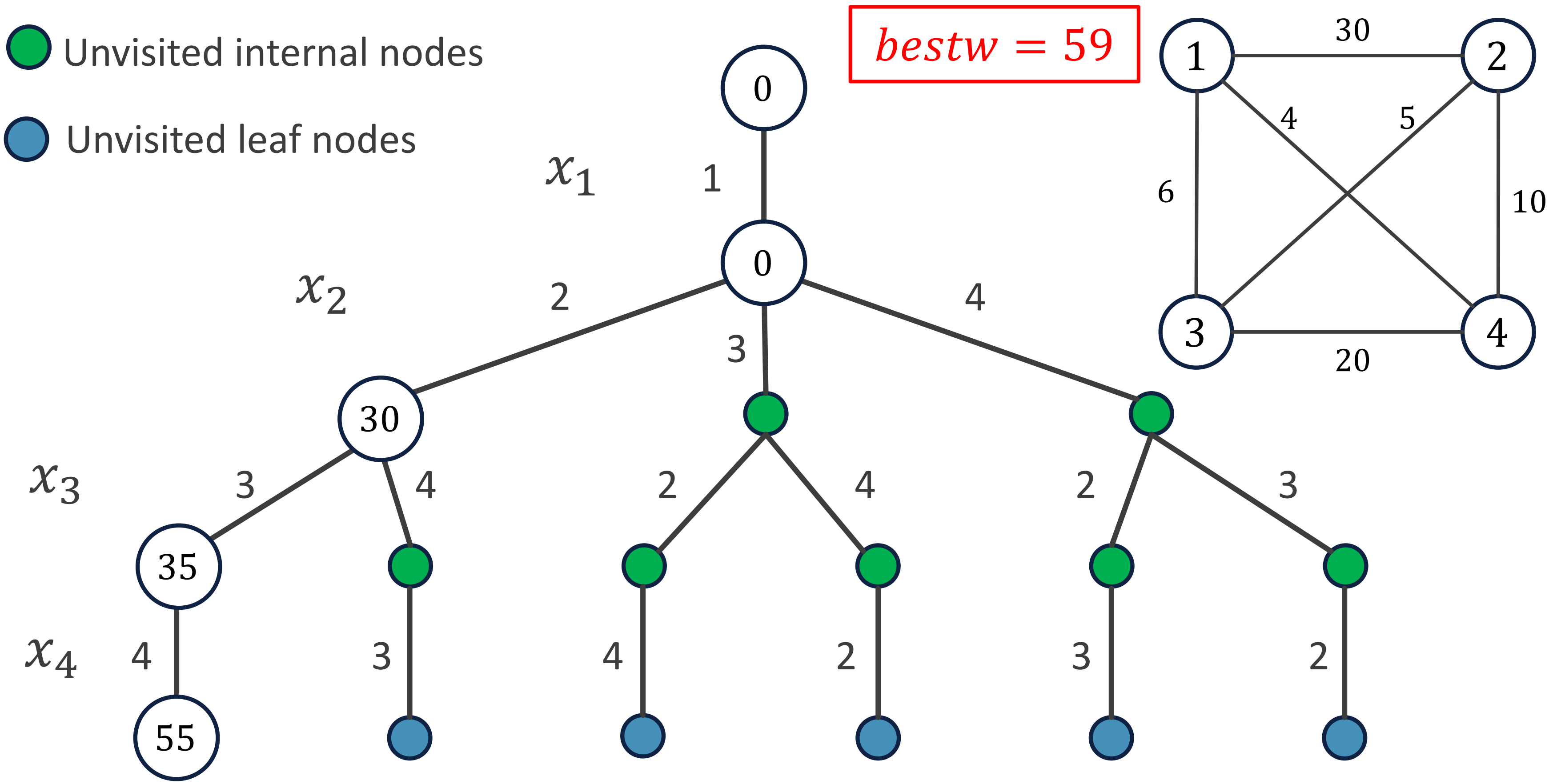


Example



Unvisited internal nodes
Unvisited leaf nodes
$bestw = \infty$
x_1
x_2
x_3
x_4
0
1
0
2
3
4
30
35
3
4
2
4
2
3
4
3
4
2
3
2
1
2
3
4
30
4
5
6
10
20

Example

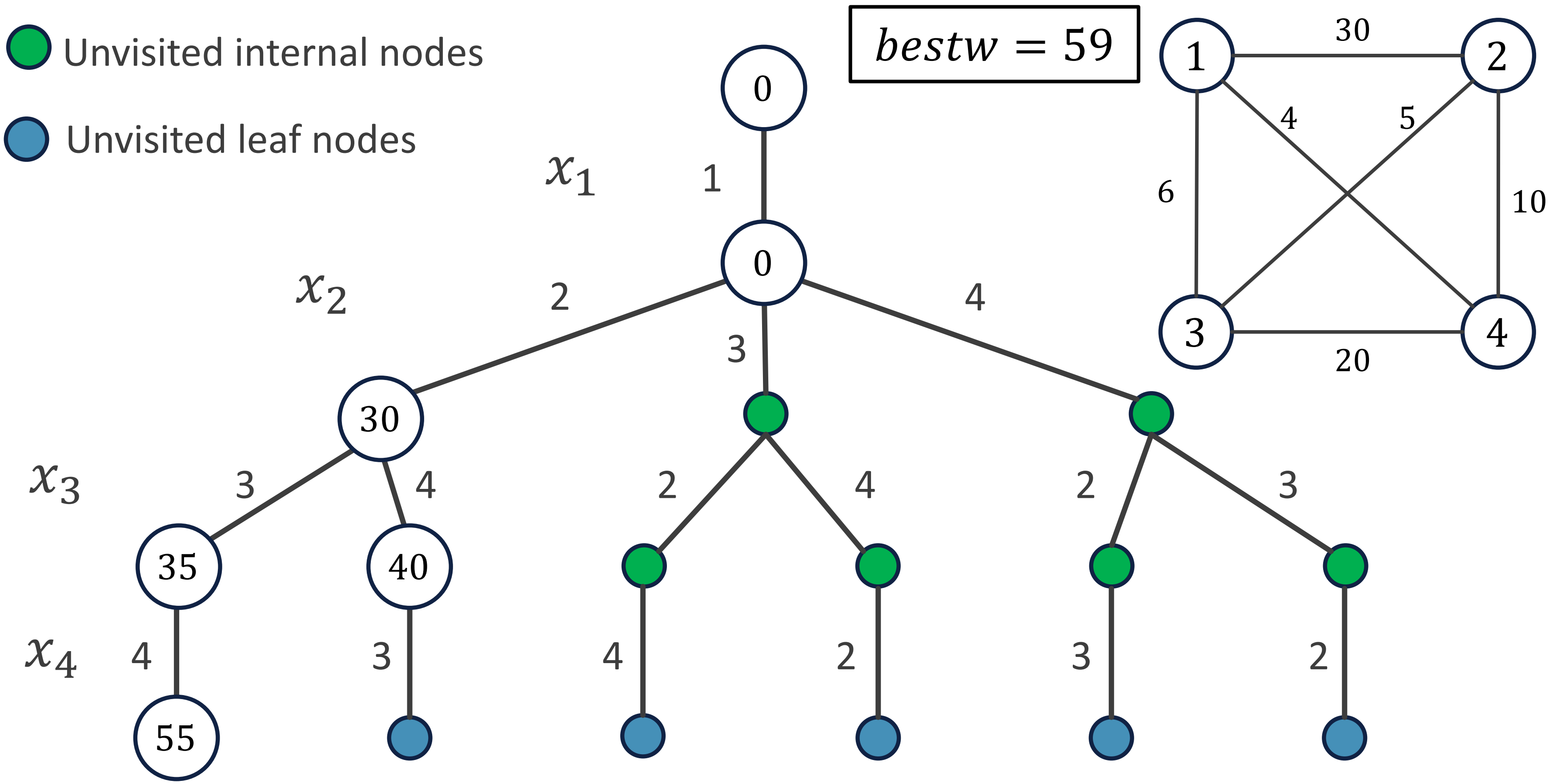


(1,2,3,4,1), 59

Example

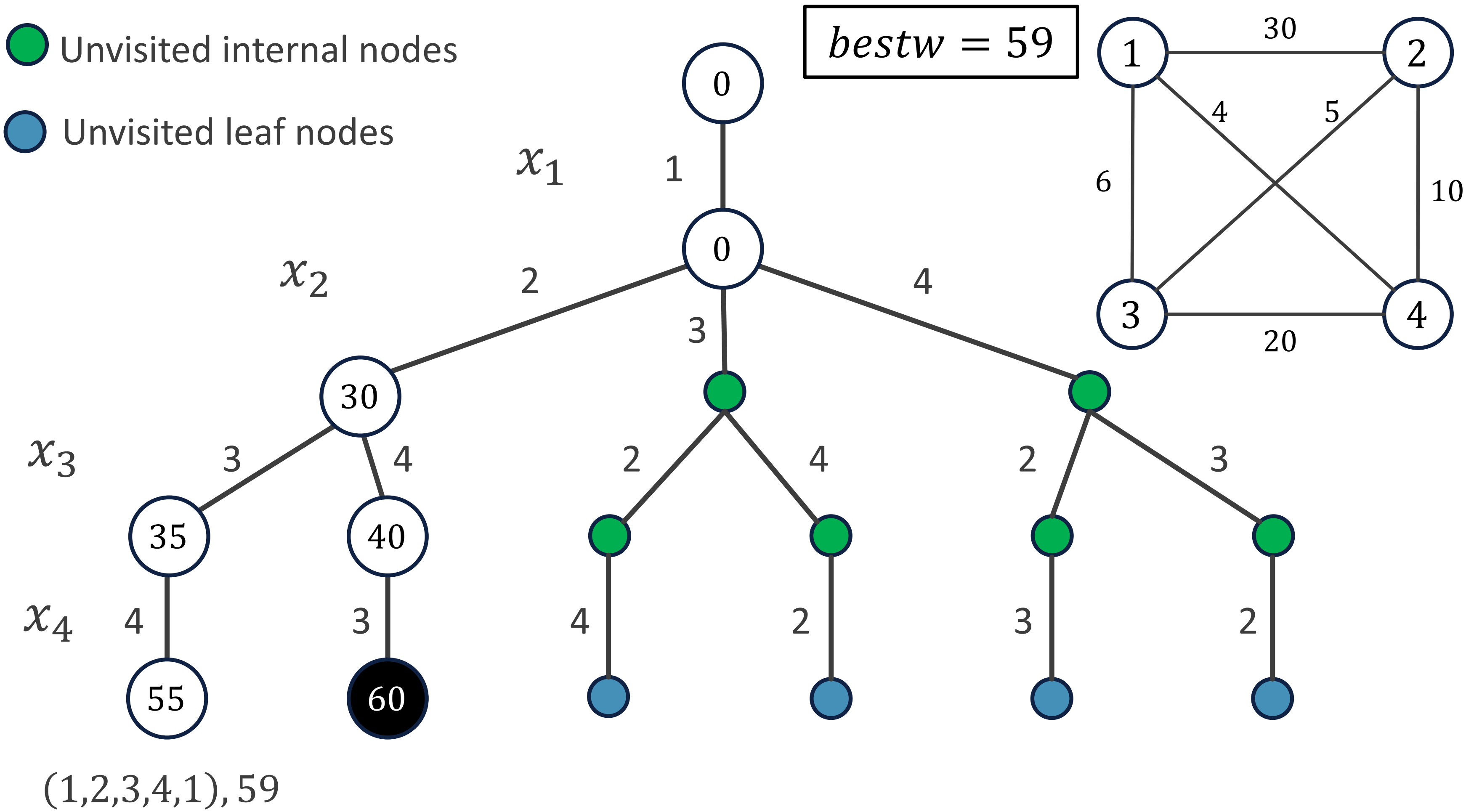
Unvisited internal nodes

Unvisited leaf nodes



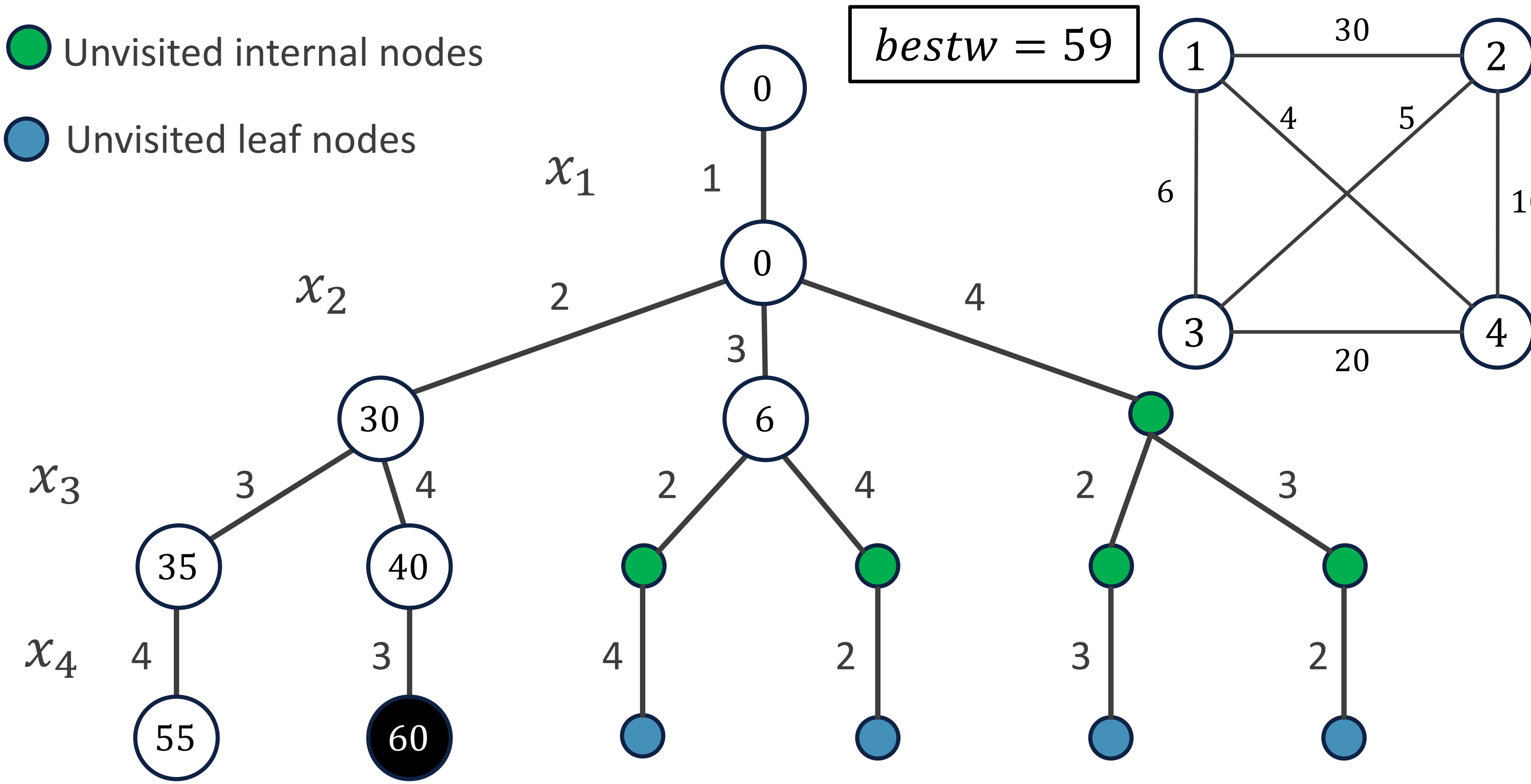
$(1,2,3,4,1), 59$

Example



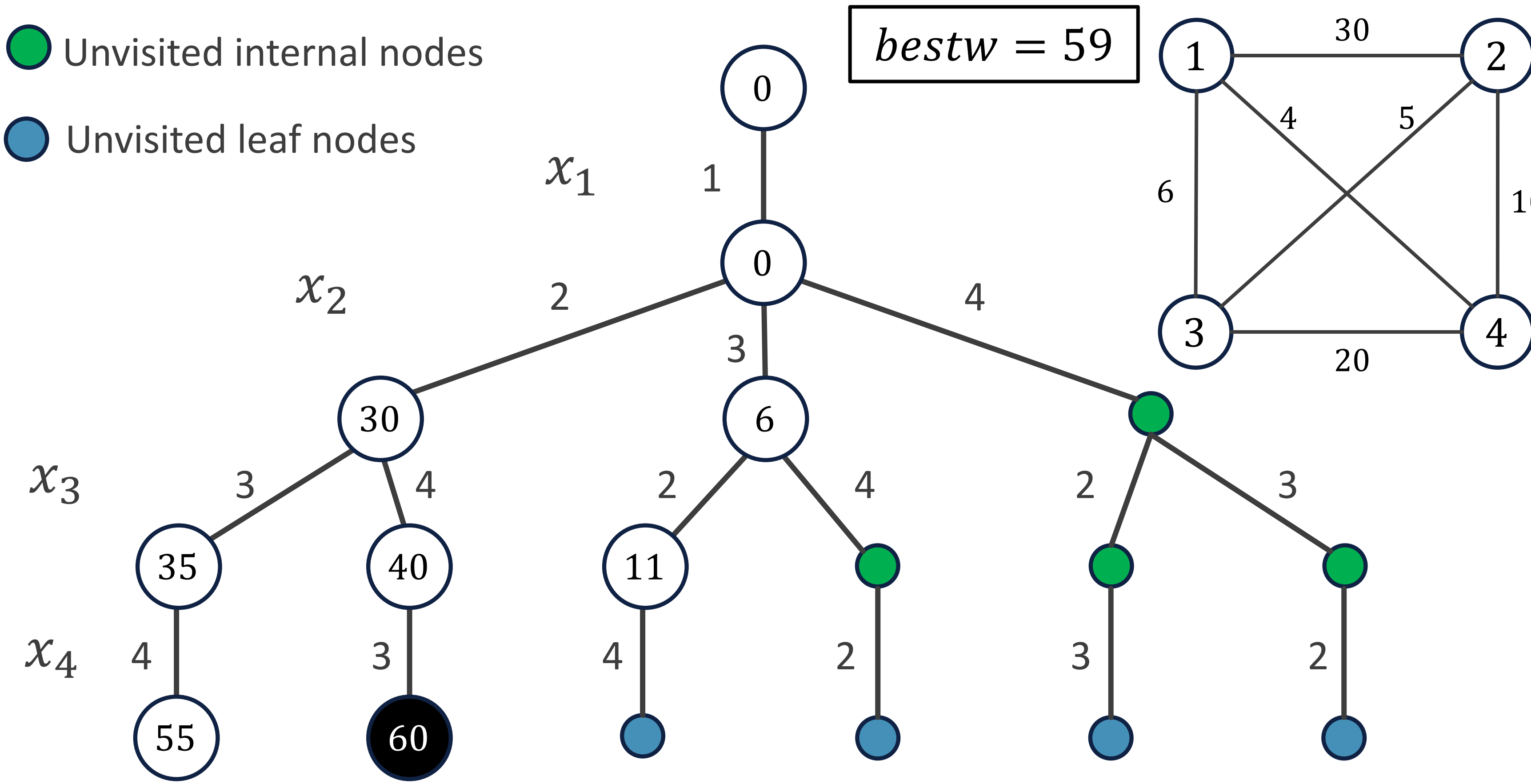
Unvisited internal nodes
Unvisited leaf nodes
$bestw = 59$
x_1
x_2
x_3
x_4
0
1
0
2
3
4
30
3
4
2
4
2
3
35
40
4
3
4
2
3
2
55
60
(1,2,3,4,1), 59
1
2
3
4
30
4
5
6
10
20

Example



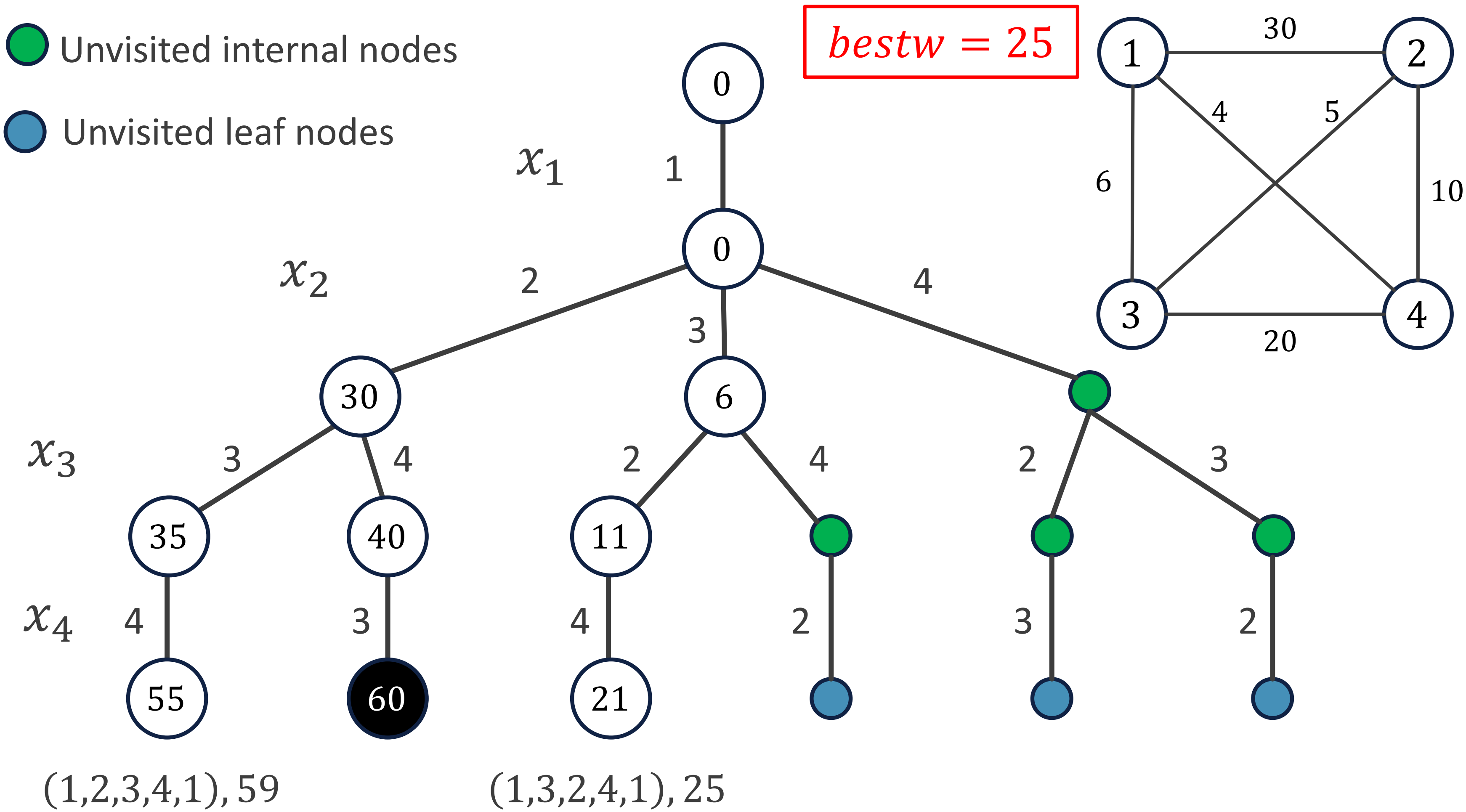
(1,2,3,4,1), 59

Example



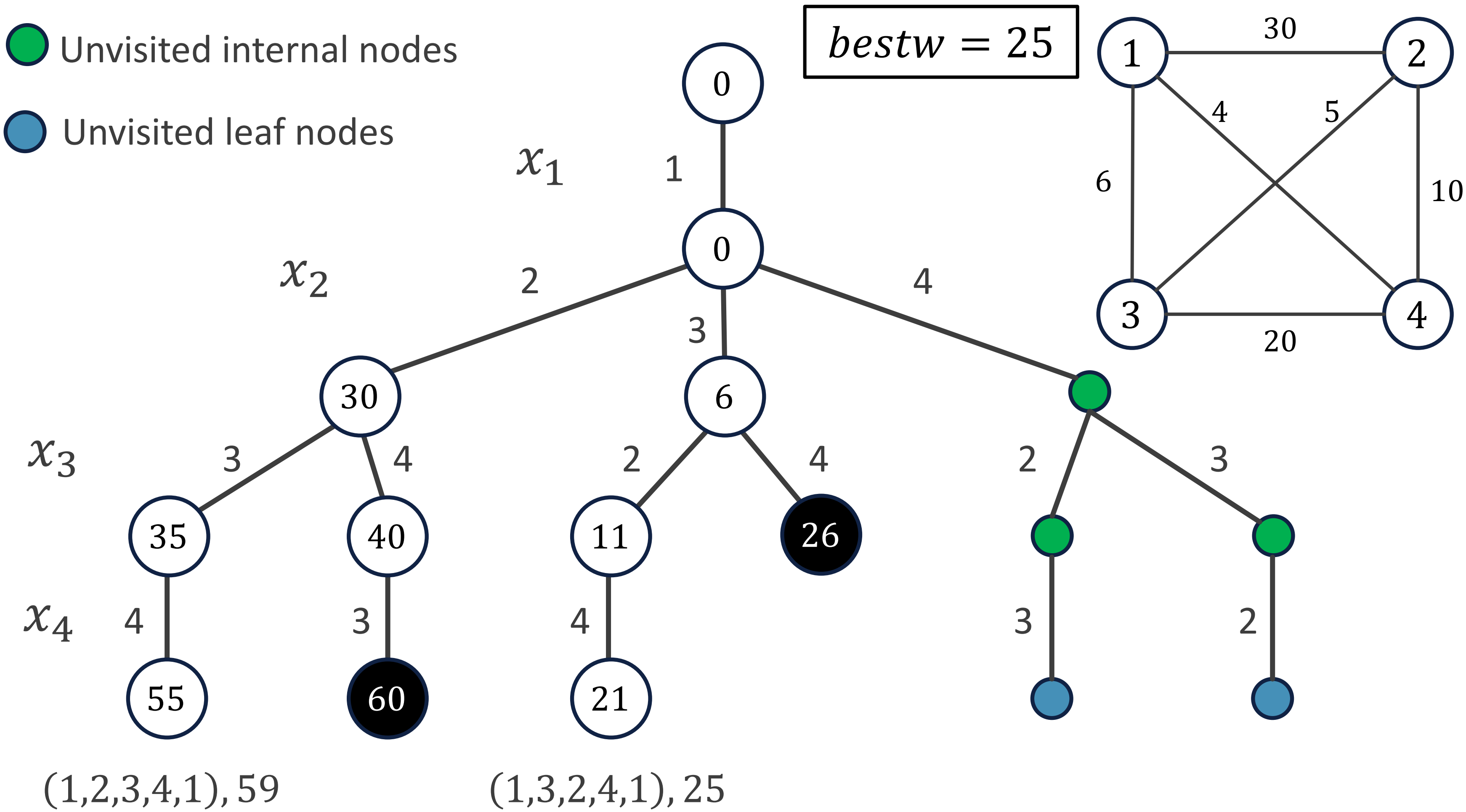
$(1,2,3,4,1), 59$

Example



Unvisited internal nodes
Unvisited leaf nodes
$bestw = 25$
x_1
x_2
x_3
x_4
(1,2,3,4,1), 59
(1,3,2,4,1), 25

Example



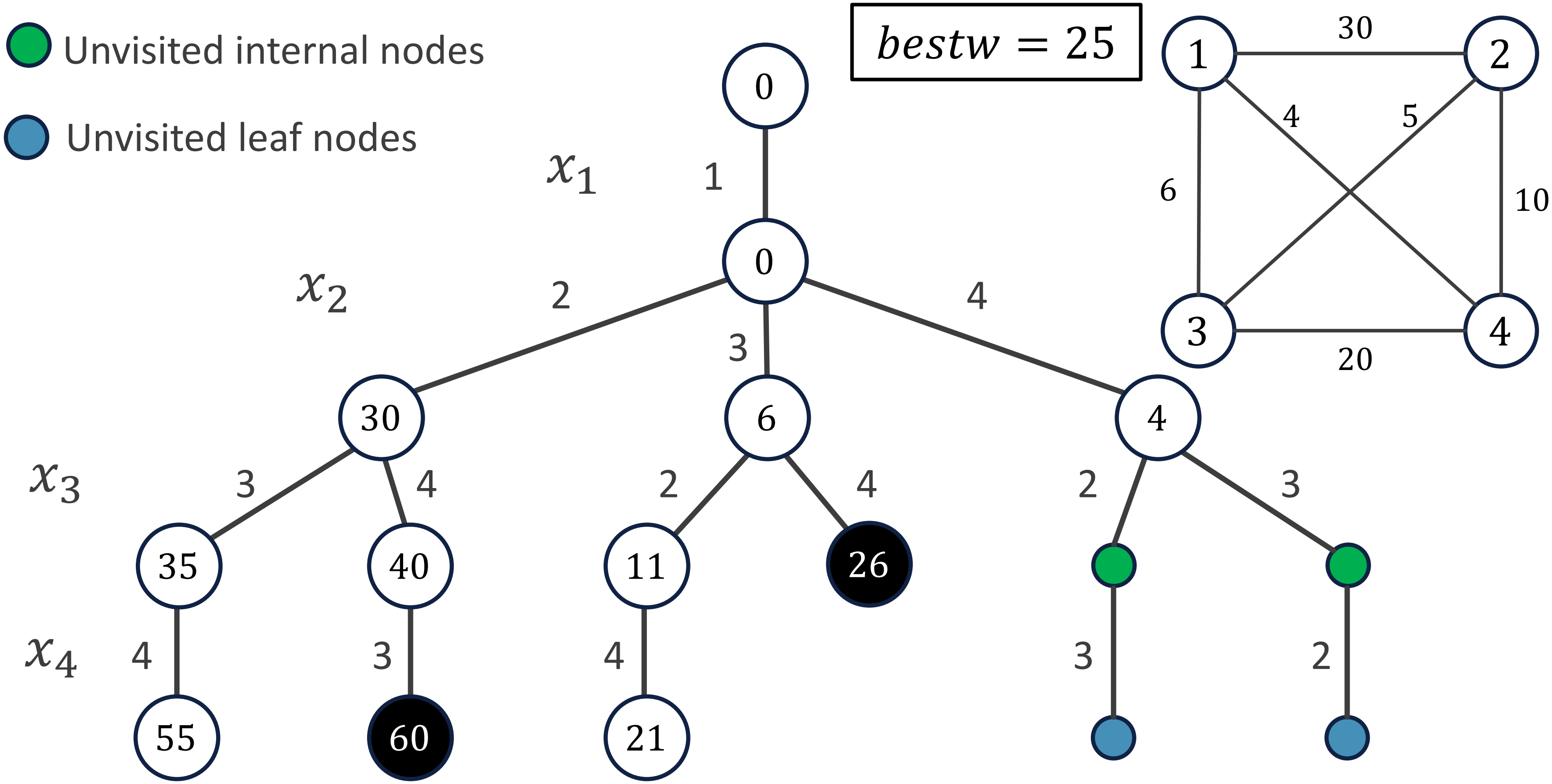
Unvisited internal nodes
Unvisited leaf nodes
bestw = 25
x_1
x_2
x_3
x_4
0
0
30
6
35
40
11
26
55
60
21
1
2
3
4
30
5
6
10
20
(1,2,3,4,1), 59
(1,3,2,4,1), 25

Example

Unvisited internal nodes

Unvisited leaf nodes

$bestw = 25$



(1,2,3,4,1), 59

(1,3,2,4,1), 25

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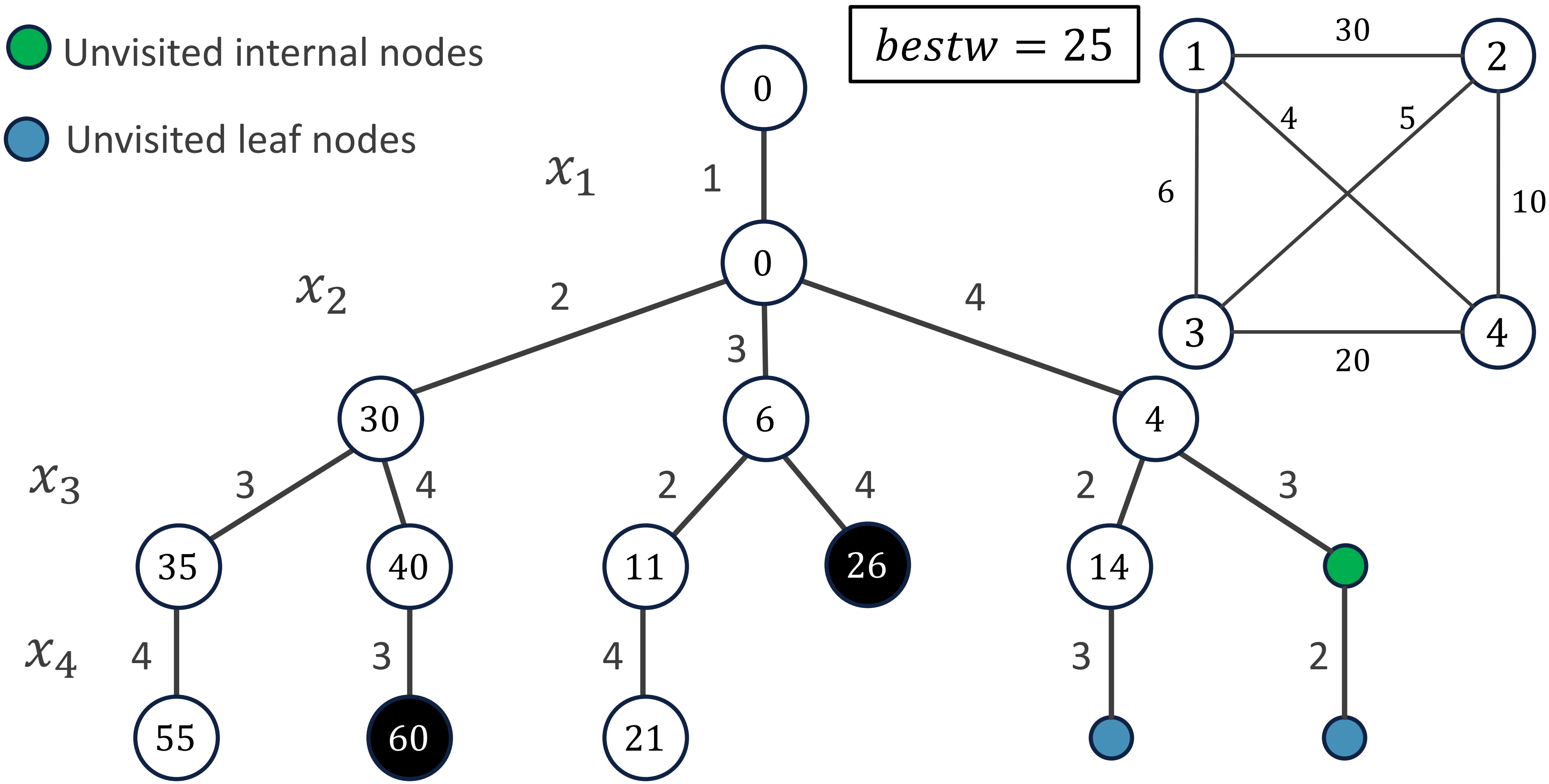
Example



(1,2,3,4,1), 59

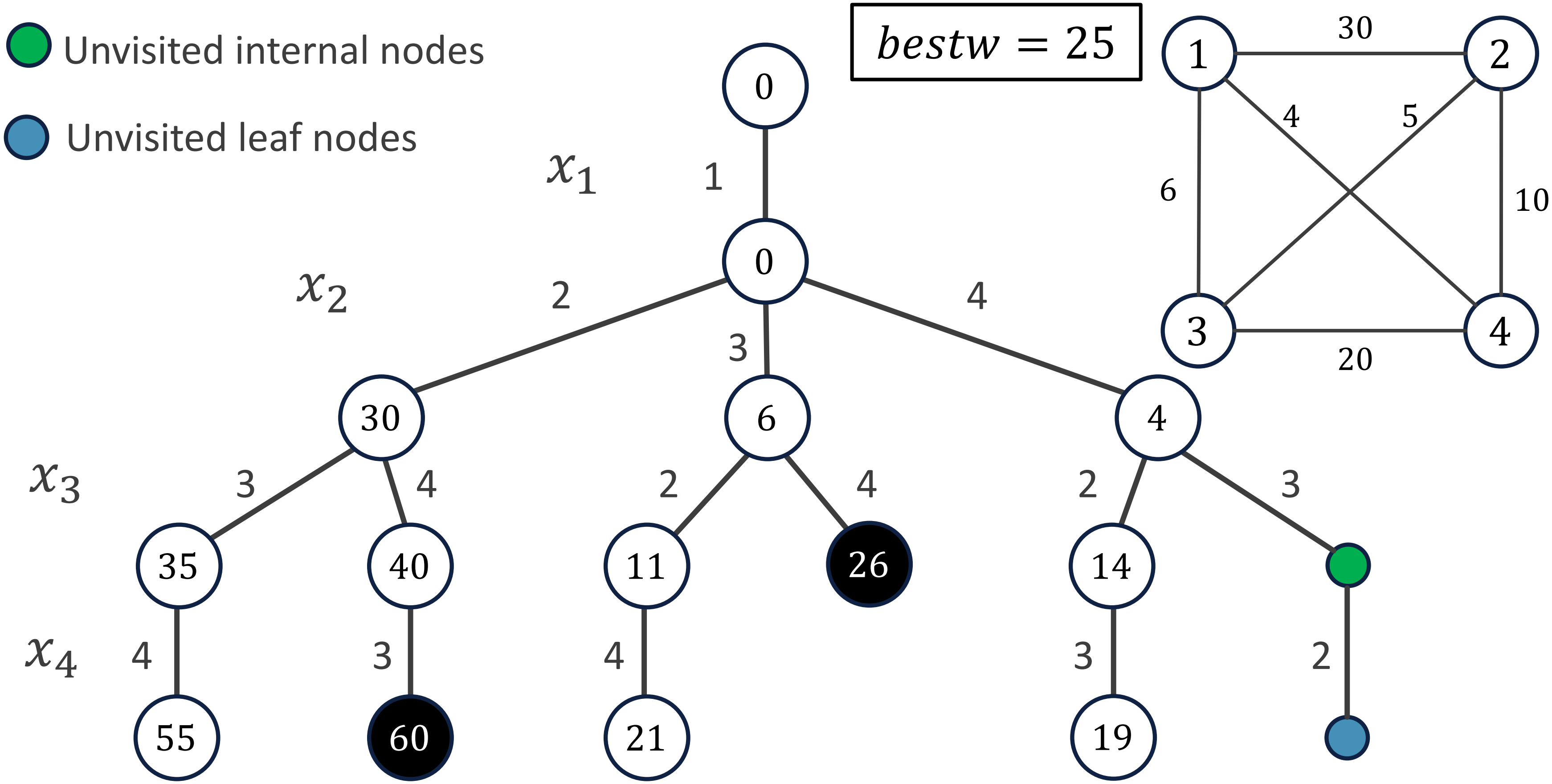
(1,3,2,4,1), 25

Example

Unvisited internal nodes

Unvisited leaf nodes

$bestw = 25$



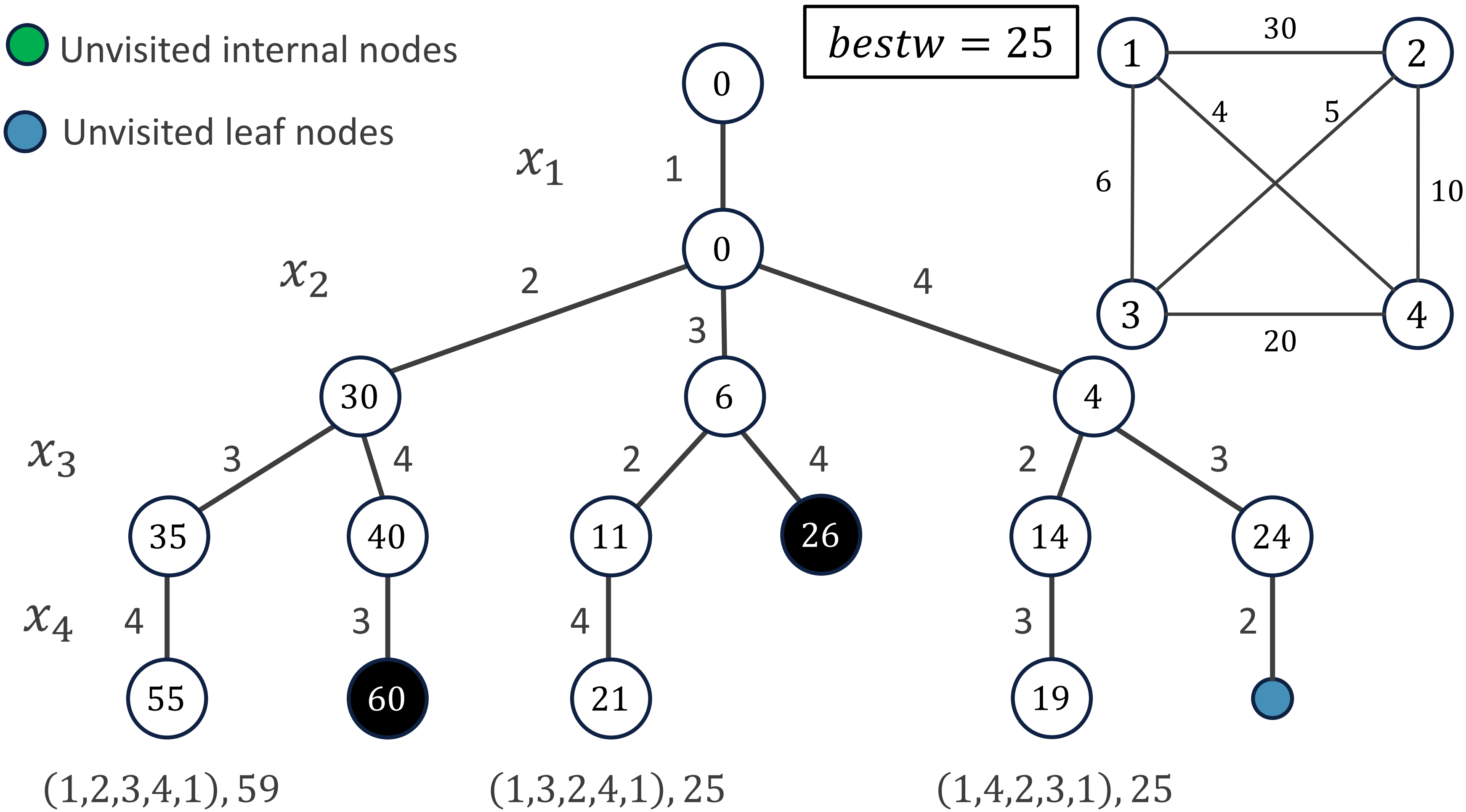
(1,2,3,4,1), 59

(1,3,2,4,1), 25

(1,4,2,3,1), 25

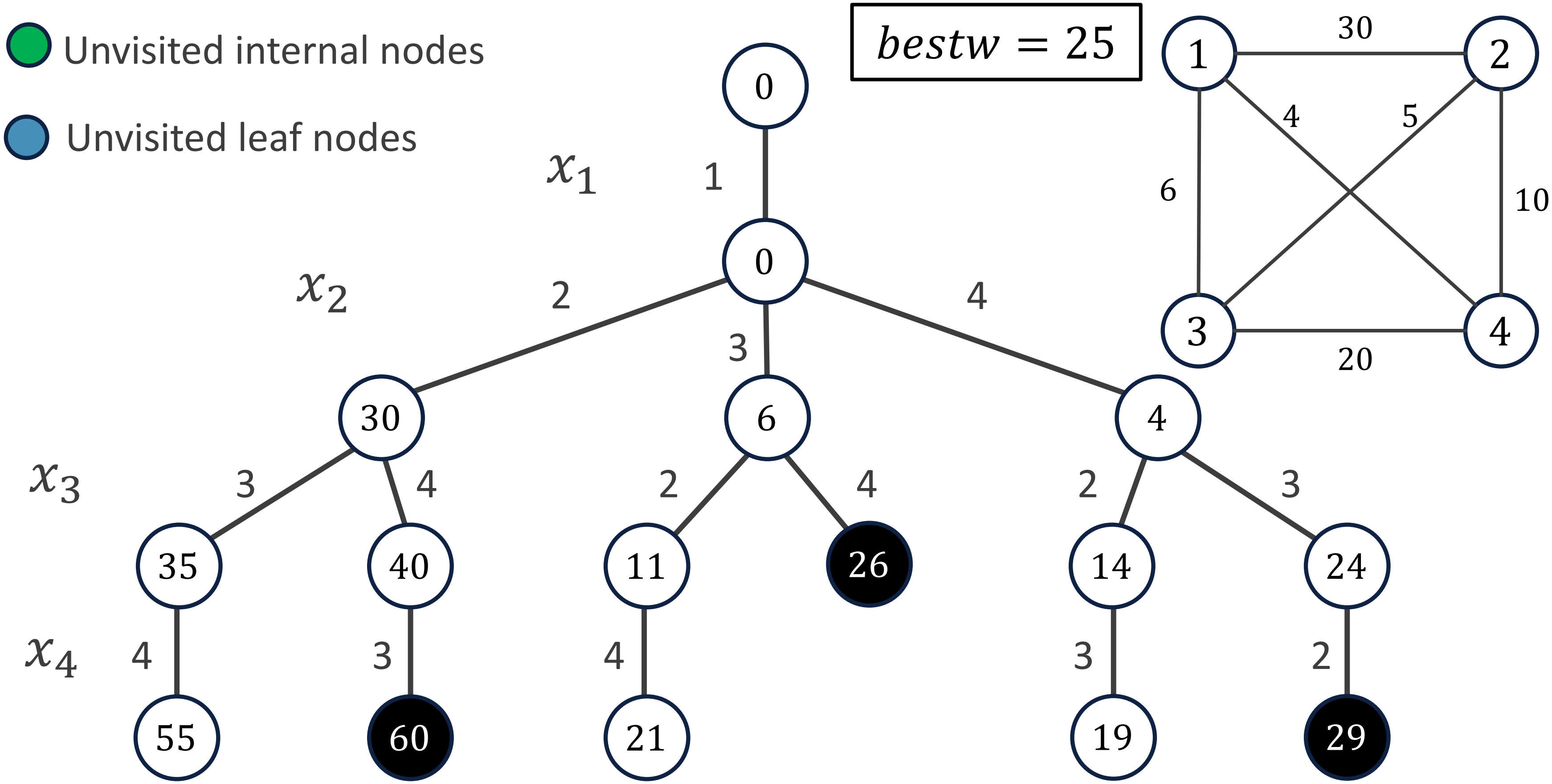
Example

- Unvisited internal nodes
- Unvisited leaf nodes



Example

- Unvisited internal nodes
- Unvisited leaf nodes



(1,2,3,4,1), 59 (1,3,2,4,1), 25 (1,4,2,3,1), 25

Pseudocode

Call BacktrackTSP(2) with initialization $x[i] = i$.

Connectivity between the last two vertices

Connectivity between the last and the first vertex

BacktrackTSP(i)

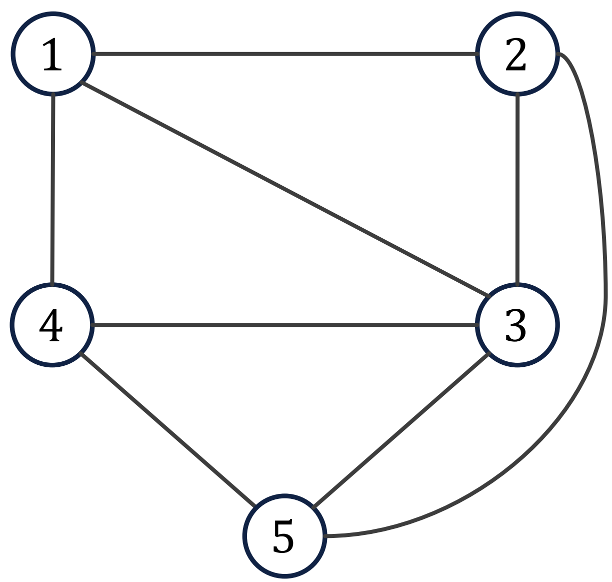
if $i = n$ **then**
    if $w[x[n-1], x[n]] \neq \infty$ and $w[x[n], 1] \neq \infty$ **then**
        if $cw + w[x[n-1], x[n]] + w[x[n], 1] < bestw$ **then**
            $bestw \leftarrow cw + w[x[n-1], x[n]] + w[x[n], 1]$
            for $j \leftarrow 1$ **to** n **do**
                $bestx[j] \leftarrow x[j]$
else for $j \leftarrow i$ **to** $n-1$ **do**
    if $w[x[i-1], x[j]] \neq \infty$ and $cw + w[x[i-1], x[j]] < bestw$ **then**
            $x[i] \leftrightarrow x[j]$
            $cw \leftarrow cw + w[x[i-1], x[i]]$
            BacktrackTSP($i + 1$)
            $cw \leftarrow cw - w[x[i-1], x[i]]$
            $x[i] \leftrightarrow x[j]$

We don't iterate to n because the last vertex is the only choice

We don't assign values to $x[i]$, instead we use permutation trick.

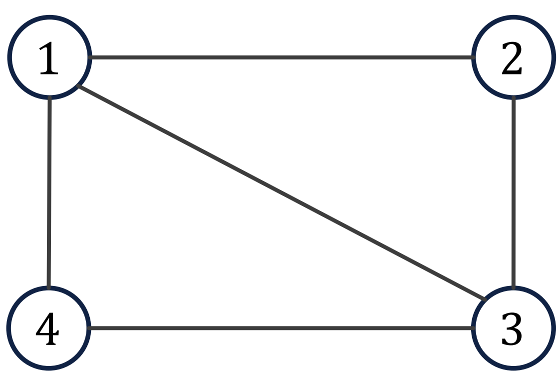
Classroom Exercise

Consider the 3-coloring problem for the given graph. Design constraint function and bounding function, and draw the pruned solution space tree to find a solution.



Classroom Exercise

- The constraint function is to check duplicated color.
- There is no bounding function for m-coloring problem.



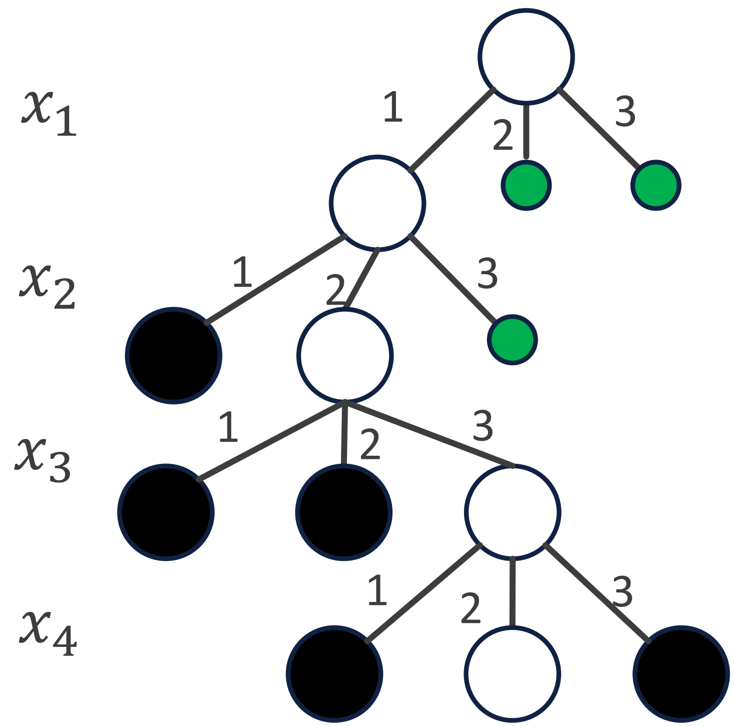


Image source: Figure 5.12, Richard E. Neapolitan, Foundations of Algorithms (5th Edition), Jones & Bartlett Learning, 2014

Conclusion

After this lecture, you should know:

- What is the difference between DFS and backtracking.
- What is a solution space tree.
- What is constraint function and bounding function.
- What kind of problems can be solved by backtracking.

Homework

Page 238-240

12.7

12.8

12.10

- For these questions, you should describe the idea of how to design constraint function and bounding function. And then write down the pseudocode.

Experiment 1

- Write a program to solve a Sudoku puzzle by filling the empty cells.
- A sudoku solution must satisfy all of the following rules:
  - Each of the digits 1-9 must occur exactly once in each row.
  - Each of the digits 1-9 must occur exactly once in each column.
  - Each of the the digits 1-9 must occur exactly once in each of the 9 3x3 sub-boxes of the grid.
- Empty cells are indicated by the character '.'.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Experiment 1

- Input: [["5","3",".",".","7",".",".",".","."],["6",".",".","1","9","5",".",".","."],[".","9","8",".",".",".",".","6","."],["8",".",".",".","6",".",".",".","3"],["4",".",".","8",".","3",".",".","1"],["7",".",".",".","2",".",".",".","6"],[".","6",".",".",".",".","2","8","."],[".",".",".","4","1","9",".",".","5"],[".",".",".",".","8",".",".","7","9"]]

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Output: [["5","3","4","6","7","8","9","1","2"],["6","7","2","1","9","5","3","4","8"],["1","9","8","3","4","2","5","6","7"],["8","5","9","7","6","1","4","2","3"],["4","2","6","8","5","3","7","9","1"],["7","1","3","9","2","4","8","5","6"],["9","6","1","5","3","7","2","8","4"],["2","8","7","4","1","9","6","3","5"],["3","4","5","2","8","6","1","7","9"]]

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Experiment 2

- 使用回溯解决石材切割问题.

谢谢

有问题欢迎随时跟我讨论

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